A Theoretical Definition and Statistical Description of Firm Size

Duarte Trigueiros

University of Macau,
Po Box 3001, Macau.
Phone: +853 3974 646, Fax: (853) 83 83 20
e-mail: fbadt@umac.mo

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Abstract

The paper offers a precise definition of firm size. Based on it, a method for estimating size is also developed and discussed. The paper first demonstrates that the assumptions of proportionality (Whittington, 1980) and lognormality of accounting variables imply the existence of size as a well-defined statistical effect. Based on this, a method for estimating size and the corresponding confidence interval is described. The paper also presents empirical evidence from sets of accounts of UK firms, overwhelmingly supporting the existence of such size effect and the feasibility of the proposed method.

Key-Words: Firm size.

Introduction

To the present date firm size remains a poorly defined concept. Where the use of size is required by theory, empirical studies typically revert to some proxy or other, such as the number of employees, Total Assets, Sales or Market Capitalisation. Conversely, the concept of firm size has also been used to proxy for numerous theoretical constructs ranging from risk to liquidity or even political costs (Ball and Foster, 1982). As a result, firm size has been interpreted in many different ways, 'allowing it to explain everything, and thus nothing, at the same time' (Bujaki and Richardson, 1997).

Yet the advantages of possessing a precise definition of firm size are evident. Variables used to proxy for firm size in empirical studies contain variability of their own. A realisation of Sales, for instance, is the result of several statistical effects of which only one, size, is desirable in this context. When proxies are used instead of size, the unwanted portion of their variability has the potential to distort predictions in statistical models.[1] For instance, size proxies are often used to deflate other variables in financial analysis and in econometric models. Presently used deflators may actually remove more than just size thus weakening the predictive power of the size-free measurement. When Market Capitalisation is used to deflate another variable it is likely that market-related effects that may be present in such variable are removed together with size. Although such removal of market effects is, in some cases, a desired feature of the model, in general it constitutes a distortion and a limitation. In contrast, a deflator that is just size would preserve all the potentially interesting effects in the deflated variable.

Also, since it is often speculated that liquidity, political costs and other theoretical constructs are related to size, the possibility of using a size measurement free from spurious influences would allow a better testing of those hypotheses.

Last but not least, a precise definition of size may contribute to dispell undue scepticism about the statistical characterisation of accounting data. Data extracted from accounting reports are often viewed as intrinsically complex, the possibility of applying clear-cut rules to such data being rejected *a priori*. As a consequence of this belief in the statistical opacity of accounting data, empirical research often lacks the level of definition that is required to draw appropriate inferences. This paper describes a statistical formulation or model capable of accounting for a highly significant proportion of the variability of accounting data in cross section. This is of great practical consequence, helping discarding the above mentioned schepticism.

The relationship between size and growth, a subject closely related to the present paper, has been studied in the context of Industrial Economics. Hart and Prais (1956) is an early study where the relative growth of British firms is examined. Singh and Whittington (1975) and Mansfield (1962) in the US are other early examples. In general, results seem to confirm the belief, expressed by Gibrat (1931), that size is broadly independent from the rate of growth of firms.[2] This, in turn, implies that accounting variables related to firm size should exhibit a proportionate (multiplicative) behaviour. In other words, rather than normally distributed, variables such as Sales or Total Assets should be viewed as broadly lognormal.

In the accounting literature, McLeay (1986) and McLeay and Fieldsend (1987), have examined the distribution of ratios formed with accounting variables that are summations of positive transactions (such as Sales, Stocks, Creditors or Current Assets) asserting that they should exhibit a proportionate behaviour. Empirical work by Trigueiros (1995) has shown that lognormality is a widespread feature of accounting data: not only summations of positive transactions, many other positive-valued items have cross-section distributions that are lognormal.[3]

Besides providing a precise definition of size, the paper develops a method allowing its estimation, based on the fact that size is a component of the variability of many different accounting numbers. It is thus possible, by building models that use numbers taken from the same set of accounts, to smooth other effects away thus isolating size. The paper also shows empirical evidence suggesting that, indeed, the effect of size is present in several accounting numbers and that it is possible, based on this, to estimate size. It should be understood that the following discussion contemplates the generative mechanism (the natural or structural form) underlying proportionality, not an observable behaviour. This is highlighted by the use of uppercase letters. In addition, the discussion is restricted to positive realisations of accounting numbers.

Proportionality and Firm Size

Any search for an estimate of size should begin by examining models that pre-suppose or implicitly recognise a size effect in variables related to the firm. Amongst these models, the best known is the simple financial ratio. Managers, Investors and analysts alike routinely use ratios of accounting numbers to compare the financial features of firms of different sizes. Examples of widely used ratios are the Return on Equity and Return on Investment ratios, the Sales Margin and Capital Turnover ratios, the Leverage and Interest Cover ratios and the Current ratio. In all of these cases and in many others, users assume that ratios are indeed capable of 'control[ling] for size so that comparisons may be made' (Barnes, 1982).

The paper is based on the premise that only existing effects can be removed. Accordingly, it should be possible to isolate whatever ratios remove by looking at the condition or conditions for their valid use. In other words, the paper explores the possibility of reversing the functional path leading to the use of ratios: instead of considering ratios as a way to remove size, the paper examines conditions for the removal of size by ratios and then derives models obeying the same conditions, capable of estimating size.

This is by no means a difficult task. Whittington (1980) has stated that the most basic condition for the validity of the ratio method is proportionality between the numerator of the ratio, Y, and the denominator, X, so that Y/X is constant. When Y is proportional to X, it may also be stated that the rate of change of Y with respect to X is constant and simlar to the ratio itself:

$$\frac{Y}{X} = \frac{dY}{dX}$$

where dY, dX are related changes or differences observed in Y and X. This formulation is the differential equivalent to Y/X = Constant. By re-arranging terms, the above becomes:

$$\frac{dY}{Y} = \frac{dX}{X}. \tag{1}$$

Equality (1) shows that the implicit condition of proportionality is *scale invariance*, whereby the relative changes in Y are equal to the relative changes in X. For instance, when comparing firms in cross-section, if the Current Assets figure is ten times larger in one firm than in another, then this should also be the case for Current Liabilities.[4]

Proportionality amongst more than two variables leads to a similar formulation. Assume first that all possible ratios formed with N variables $X_1, \cdots X_N$ (eg., X_1/X_2, X_1/X_3, X_2/X_3 and so on), are proportional:

$$\begin{aligned} X_1/X_2 &= \text{Constant}_{1,2} \\ X_1/X_3 &= \text{Constant}_{1,3} \\ &\vdots \\ X_2/X_3 &= \text{Constant}_{2,3} \\ &\vdots \end{aligned}$$

i.e., for any possible pair $k \leq N, l \leq N$ it is verified that

$$X_k/X_l = \text{Constant}_{k,l} \tag{2}$$

Similarly to the reasoning leading to (1), it may be concluded from (2) that relative changes in all N variables are the same, i.e.,

$$\frac{dX_1}{X_1} = \frac{dX_2}{X_2} = \cdots = \frac{dX_N}{X_N} \tag{3}$$

In fact, since

$$\frac{dX_1}{X_1} = \frac{dX_2}{X_2} \text{ and } \frac{dX_1}{X_1} = \frac{dX_3}{X_3} \text{ it follows that } \frac{dX_2}{X_2} = \frac{dX_3}{X_3} \text{ and so on.} \tag{4}$$

It is most likely that the above generalisation of proportionality applies to real-world accounting numbers, specifically to those components of commonly used ratios. Notice first that the validity of the ratio method rests on the validity of several widely used ratios. This seems to match assumption (2). Also, most of the items used to form a specific ratio are used as well to form other ratios. Sales, for instance, is used by analysts to form Margin, Turnover and Profitability ratios, together with variables such as Total Assets, Cost of Goods Sold or Gross Profits. Equality (1) thus requires, closely following (4), that relative differences or changes observed in those variables be the same as in Sales. Total Assets is routinely used to form Leverage ratios thus extending this requirement to Net Worth. Since the subtraction of two variables preserves common relative changes, Total Liabilities (Total Assets minus Net Worth) also has to obey the same requirement. Cost of Goods Sold, in turn, extends it to Inventory via the popular Inventory Turnover ratio; Net Worth, being the denominator of Return on Equity, extends it to Net Income, thence to Dividends and Interest Expense. Indeed, it may be difficult to identify aggregate items from accounting reports that analysts never combine with other items to form more than one ratio, or that are not forced by accounting identities to change at the same rate as other items.

Now consider two realisations, B and A, of the same N variables $X_1, \cdots X_N$ so that, in (3), changes or differences between B and A are expressed as

$$\begin{aligned} dX_1 &= X_{1_B} - X_{1_A} \\ &\vdots \\ dX_N &= X_{N_B} - X_{N_A} \end{aligned}$$

The above, together with the fact that (3) may be written as a set of equations, implies

$$\begin{aligned} X_{1_B} &= X_{1_A}(1 + r_{BA}) \\ &\vdots \\ X_{N_B} &= X_{N_A}(1 + r_{BA}) \end{aligned}$$

where r_{BA} is the relative difference by which any item in B differs from the corresponding item in A. Therefore, structurally at least, if one item is larger in B than in A, it follows that any other item will also be larger in B than in A and conversely. It is thus possible to say without ambiguity that set B is, as a whole, larger or smaller than set A. This possibility of comparing the size of two sets of accounts make it possible to rank by size, without ambiguities other than those created by identical sets, any collection $A, B, \ldots, J, \ldots$ of sets of accounts. Then, by assigning to each set in the ranked collection one unique ordinal number denoting its position, a discrete, *ordinal size* variable is created.

Likewise, it is possible to construct a *continuous size* measurement by using a suitable *unit size* set, $\mathcal{X}_1, \cdots \mathcal{X}_N$,[5] and then measuring the $S_J = 1 + r_J$ specific to the J^{th} set of accounts against this unit size. Such measurement shows how many times a set of accounts is larger or smaller than the unit. If ratios are valid, therefore, it should be possible (using statistical sampling and modelling techniques) to estimate the size of a set of accounts and the corresponding confidence interval.

The above is just one possible way of demonstrating that a precise definition of firm size is inherent to proportionality (and therefore to the valid use of ratios). The operation whereby a realisation of N variables stems from scaling up or down other realisation so that differences are invariant in relative terms, obviously implies the existence of another variable, size, capable of scaling different realisations by the same factor.

The Estimation of Firm Size

Consider, as before, sets of accounts $A, B, \ldots, J, \ldots$ each of them containing items $1, \cdots, N$. Corresponding items from different sets are viewed as realisations of accounting variables $X_1, \cdots X_N$. For example, X_5 may contain Current Assets values taken from sets $A, B, \cdots, J, \cdots$ whereas the J^{th} set of accounts contains N items of which Current Assets would be the 5^{th}. According to the reasoning presented in the previous section, the k^{th} item from the J^{th} set of accounts is described as

$$X_{kJ} = \mathcal{X}_k \, S_J \tag{5}$$

where the constant $\mathcal{X}_k$ is the value of item k in an appropriate unit set and $S_J = 1 + r_J$, the size of the J^{th} set of accounts, is the number of times any item in J is larger or smaller than the corresponding item in the unit set. This is an obvious description of proportionality whereby each realisation of $X_1, \cdots, X_N$ stems (amongst other effects) from applying the same scale to variable-specific level values. Indeed, so long as $\mathcal{X}_k$ is independent from the set of accounts considered and S_J is likewise independent from the variable considered, (3) is verified and, of course, S_J is removed from the ratio measurement.

Now consider random variables $x_1, \cdots, x_N$ which are cross section realisations consistent with the corresponding $X_1, \cdots, X_N$. The paper assumes that these random variables obey the simplest form of multiplicative behaviour, lognormality, so that $\log x_{kJ} \equiv \mathcal{N}(\mu, \sigma^2)$ or, in the case of three-parametric lognormality, $\log(x_{kJ} - \delta_k) \equiv \mathcal{N}(\mu, \sigma^2)$ where δ_k is the threshold of the distribution (Aitchison and Brown, 1957). From (5),

$$\log x_{kJ} = \log(\mathcal{X}_k S_J) + \varepsilon_{kJ}$$

where ε_{kJ} is an error term. Thus,[6]

$$\log x_{kJ} = \mu_k + \varsigma_J + \varepsilon_{kJ}. \tag{6}$$

As before, μ_k is the statistical effect of variable k on $\log x_{kJ}$ whereas ς_J is the effect of the J^{th} set of accounts. It should be noticed that μ_k is a *fixed* effect whereas ς_J is a *random* effect.[7] Thus (6) may best be interpreted and estimated as a *mixed effects* model. The random effect, $\varsigma_J = \log(1 + r_J)$, is a continuously compounded r_J. It measures the *logarithmic size* of set J. The fixed effect is the *logarithmic unit* or level of variable k ($\mu = \log \mathcal{X}_k$).

In is easy to show that, unless the two components of a ratio are lognormal, the ratio formed from them will not be size-free. Ratios remove common effects because a quotient of two exponentiations is the exponentiation of a difference. Thus any effect that exists both in the numerator and in the denominator of a ratio cancels out.

In short, proportionality without a broadly multiplicative behaviour of components would preclude validly used ratios and this limitation would probably show up in the form of a reduced applicability or scope. Indeed, ratios are popular basically because financial variables are broadly multiplicative.

Notice also that, in (6), nothing is said about the role of μ_k. In fact, one of the two parameters above may be selected arbitrarily and the other will ensue from the requirement that $\varepsilon_{kJ} \equiv \mathcal{N}(0, \sigma^2)$. For instance, when it is decided that μ_k takes its usual role as the expected value of $\log x_{kJ}$, then an estimated $\mathcal{X}_k$ is the geometric mean of variable k and the logarithmic size has an expected value of zero. In other cases, for instance in the context of time series, it may make sense to let $\mathcal{X}_k$ assume the initial value of x_{kJ} so that ς_J is a growth rate.

Suppose that, given a cross section sample of accounting numbers, μ_k is estimated as m_k, the average $\log x_k$. Since, inside each set of accounts (say, set J), the effect of ς_J is the same for all items, this parameter may be estimated as s_J, the expected value of $\log x_{kJ} - m_k$. In other words, given the J^{th} set of accounts containing N items, it is possible to estimate ς_J by randomly selecting a subset $x_1, \cdots, x_M$ and then averaging values

$$\begin{aligned} \log x_1 - m_1 &= s_J + \varepsilon_1 \\ &\vdots \\ \log x_M - m_M &= s_J + \varepsilon_M \end{aligned}$$

so that

$$s_k = \frac{1}{M} \sum_{i=1}^{M} (\log x_i - m_i) - \frac{1}{M}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_M).$$

Since any source of variability common to all the $\log x_i$ is incorporated into s_i, no significant common variability remains in the ε_i. Therefore, although correlation amongst individual ε_i may still exist, independence must be verified amongst groups of these deviates. As a consequence, the second term above should tend to zero, leading to

$$s_k \approx \frac{1}{M} \sum_{i=1}^{M} [\log x_i - m_i] \tag{7}$$

for large enough M. In case significant thresholds exist in the distributions of the x_i, those should be accounted for before averaging:

$$s_k \approx \frac{1}{M} \sum_{i=1}^{M} [\log(x_i - \delta_i) - m_i] \, .$$

Exact confidence intervals for s_J can be obtained, the corresponding standard errors being t-distributed with $M - 1$ degrees of freedom.

The practical difficulty in estimating ς_J stems from the fact that the x_i cannot be randomly drawn from all possible items in a set of accounts. In the first place, for firms of the same size, clusters of items are significantly correlated, denoting the existence of correlation amongst the corresponding ε_i. Where some of the ε_i are correlated, the number of items required to smooth error terms away increases. Also, items such as Earnings have negative values and cannot be transformed into logarithms. This leads to the need to pre-select items before randomisation, thus introducing a potential bias in the estimation of size.

Empirical Results

This section presents an exploratory data analysis supporting the hypotheses that size may be described as a statistical effect consistent with (6) and that size may be estimated from accounting numbers as described in (7).

Data used in the paper are taken from the Micro-EXSTAT database for five consecutive years (1983-1987). Following Sudarsanam and Taffler (1985), 14 industries and 18 variables are used to form single- or multiple-industry samples (tables 1 and 2). Only UK firms are selected. The number of firms in single-industry samples ranges from a minimum of 13 (Leather, 1983) to a maximum of 145 (Electronics, 1986). The number of firms in the multiple-industry samples ranges over the years from 94 (Food, 1983) to 273 (Engineering, 1986). Also included in the analysis is the number of employees (N) which allows the comparison with a non-accounting variable exhibiting similar statistical characteristics.

The cross section distributions of these samples are well known (Trigueiros, 1995). Most of them are approximated by the two- or three-parametric lognormal. In a few samples, namely in the Electronics and Food Manufacturing industries, the existence of clusters of firms leads to the rejection of lognormality. All variables exhibit positive kurtosis to some extent, even after the logarithmic transformation.

The observation of variance-covariance matrices of logarithms of accounting variables offers an initial view of the role and importance of size as a statistical effect. Inside each industry, the cross-section variances and co-variances of all of these variables are remarkably similar, with no negative or zero co-variances and all magnitudes above a threshold.

Table 3 illustrates how such matrices look like. The same regularity is observed for all the other industries and years considered. Matrices such as those in table 3 are only possible where there exists a sizable source of common variability. It seems as though the logarithms of accounting variables are just replications of the same variable, size, to which a small amount of specific randomness has been superimposed.

This empirical observation is consistent with (6) but it goes beyond it, showing that the common variability introduced by the effect of firm size is dominant. Indeed, this type of variance-covariance matrices is possible only when, in (6) the variability of ς_J is huge when compared with that of ε_{kJ}. Thus in cross-section, size explains a remarkably large proportion of the variability of accounting numbers.

Formal tests of the above may be obtained by comparing the variability of accounting numbers within sets of accounts with the variability between sets. If there exists a

PROCESSING:	Building Materials	Metallurgy
	Paper and Packing	Chemicals
ENGINEERING:	Electrical	Industrial Plants
	Machine Tools	Electronics
	Motor Components	
TEXTILES:	Clothing	Wool
	Miscellaneous Textiles	Leather
FOOD:	Food Manufacturers	

Table 1: List of the industrial groups examined by this study.

TA	Total Assets	NW	Net Worth
FA	Fixed Assets	DEBT	Long Term Debt
D	Debtors	C	Creditors
CA	Current Assets	CL	Current Liabilities
I	Inventory	TC	Total Capital Employed
WC	Working Capital	TD	Total Debt
EB	Earnings bf. Interest & Tax	W	Wages
OPP	Operating Profit	QA	Sales
FL	Gross Funds fr. Operations	N	Number of Employees

Table 2: Variables and abbreviations used in the paper.

	S	NW	W	I	CA	FA	EB	S	NW	W	I	CA	FA	EB
S	0.35							0.21						
NW	0.32	0.39				Paper and Pack.		0.22	0.26				Leather	
W	0.35	0.36	0.41					0.19	0.21	0.18				
I	0.33	0.34	0.38	0.40				0.19	0.21	0.18	0.19			
CA	0.33	0.32	0.35	0.35	0.34			0.21	0.24	0.20	0.20	0.23		
FA	0.32	0.36	0.37	0.34	0.33	0.40		0.19	0.21	0.18	0.17	0.19	0.20	
EB	0.35	0.35	0.36	0.35	0.34	0.35	0.41	0.26	0.29	0.24	0.23	0.27	0.24	0.28
S	0.17							0.55						
NW	0.17	0.20				Clothing		0.57	0.73				Food Manuf.	
W	0.18	0.18	0.23					0.55	0.64	0.64				
I	0.19	0.20	0.20	0.23				0.59	0.68	0.63	0.73			
CA	0.17	0.18	0.18	0.20	0.19			0.56	0.63	0.58	0.65	0.61		
FA	0.17	0.19	0.21	0.20	0.17	0.26		0.57	0.69	0.65	0.65	0.61	0.71	
EB	0.17	0.17	0.18	0.18	0.17	0.19	0.31	0.59	0.71	0.64	0.70	0.66	0.69	0.73

Table 3: Variance-covariance matrices of four industries in 1983.

common source of variability in numbers from the same set, then an analysis of variance (ANOVA) should reject the null hypothesis that numbers taken from the same set of accounts add as much variability to the total as they would without the partition into sets of accounts.

Three tests are conducted: the test of one effect at a time, the test of the two effects together in a mixed model and finally this same test but where size is previousely accounted for. Results of the first test show that, indeed, numbers taken from the same set of accounts exhibit significantly less variability than across sets. For all industries and years in the sample, within–set mean squares are significantly smaller than the corresponding between–sets mean squares. The proportion of variability explained by the fact that numbers belong to a given set of accounts (known as the Eta squared) is high, ranging from 55% to 75%. In contrast, when comparing the variability within variables with that between them, the same ANOVA method yields Eta squared proportions clearly smaller, ranging from 19% to 42%.

The second test shows that, when the two effects are incorporated into the same analysis, their level of significance is not affected. A *components of variance* algorithm is used to specify a mixed model consistent with (6) where the two main effects k (the variable) and J (the set of accounts) are tested against the residual sum of squares. Table (4) shows the results for 1987. Remarkably, the proportion of variablilty explained by the models is higher than 80% for all industries and years considered.

The same model and algorithm was then used to test the significance of k and J, but after the effect of an estimated size had been removed. If, indeed, it is possible to estimate size as suggested in (7), then the use of size-adjusted variables should lead to the complete loss of significance of J. Table (5) shows the results of this test for 1987. The removal of size is obtained by subtracting from logarithms of observations the corresponding s_J; the exact procedure used to estimate ς_J is explained below in (8).

Results of the three tests for the years of 1983–1986 are similar to those for 1987. They provide a formal test of the model proposed in the paper, showing, not only that the model explains the variability of accounting numbers, but also that little else remains to be explained after removing the variability described by (6).

This section now turns to the problem of estimating size. Although the paper points out that, in (7), the sample of M numbers taken from the same set of accounts should be random, the small number of variables available in this case has imposed a deterministic selection instead. Moreover, not all accounting variables are equally adequate to estimate size, the best variables being those with small variability of their own and uncorrelated

Industry	Effect	Sum of Squares	df	Mean Square	F	Sig.
Build. Mat.	VARIABLE	65.448	7	9.350	506.963	.000
	SET	105.648	37	2.855	154.824	.000
	ERROR	4.777	259	1.844E-02		
Metallurgy	VARIABLE	104.815	7	14.974	115.037	.000
	SET	109.191	32	3.412	26.215	.000
	ERROR	28.896	222	.130		
Paper	VARIABLE	106.129	7	15.161	360.981	.000
	SET	184.494	57	3.237	77.065	.000
	ERROR	16.422	391	4.200E-02		
Chemicals	VARIABLE	118.931	7	16.990	366.179	.000
	SET	142.513	54	2.639	56.880	.000
	ERROR	17.353	374	4.640E-02		
Electrical	VARIABLE	73.775	7	10.539	297.094	.000
	SET	127.500	45	2.833	79.870	.000
	ERROR	11.174	315	3.547E-02		
Ind. Plant	VARIABLE	39.106	7	5.587	98.670	.000
	SET	74.609	22	3.391	59.897	.000
	ERROR	8.719	154	5.662E-02		
Machine T	VARIABLE	41.352	7	5.907	153.205	.000
	SET	45.879	24	1.912	49.577	.000
	ERROR	6.478	168	3.856E-02		
Electronics	VARIABLE	283.062	7	40.437	956.648	.000
	SET	479.873	142	3.379	79.948	.000
	ERROR	41.720	987	4.227E-02		
Motor Co	VARIABLE	46.350	7	6.621	157.102	.000
	SET	106.607	28	3.807	90.336	.000
	ERROR	8.261	196	4.215E-02		
Clothing	VARIABLE	70.603	7	10.086	260.627	.000
	SET	68.999	49	1.408	36.387	.000
	ERROR	13.274	343	3.870E-02		
Wool	VARIABLE	31.560	7	4.509	112.780	.000
	SET	39.411	19	2.074	51.886	.000
	ERROR	5.317	133	3.998E-02		
Mi Textiles	VARIABLE	65.033	7	9.290	151.845	.000
	SET	220.749	37	5.966	97.513	.000
	ERROR	15.663	256	6.118E-02		
Leather	VARIABLE	24.710	7	3.530	84.900	.000
	SET	59.491	15	3.966	95.387	.000
	ERROR	4.366	105	4.158E-02		
Food	VARIABLE	247.543	7	35.363	556.627	.000
	SET	590.335	113	5.224	82.230	.000
	ERROR	49.809	784	6.353E-02		

Table 4: Tests of between-subjects Effects for 1987, mixed model where SET is the random effect and VARIABLE is the fixed effect.

Industry	Effect	Sum of Squares	df	Mean Square	F	Sig.
Build. M	VARIABLE	65.448	7	9.350	506.963	.000
	SET	.162	37	4.369E-03	.237	1.000
	ERROR	4.777	259	1.844E-02		
Metallurgy	VARIABLE	99.900	7	14.271	109.365	.000
	SET	.474	30	1.579E-02	.121	1.000
	ERROR	27.404	210	.130		
Paper	VARIABLE	101.579	7	14.511	372.444	.000
	SET	.200	54	3.696E-03	.095	1.000
	ERROR	14.728	378	3.896E-02		
Chemicals	VARIABLE	114.710	7	16.387	347.970	.000
	SET	.194	51	3.805E-03	.081	1.000
	ERROR	16.812	357	4.709E-02		
Electrical	VARIABLE	73.775	7	10.539	297.094	.000
	SET	.191	45	4.242E-03	.120	1.000
	ERROR	11.174	315	3.547E-02		
Ind. Plant	VARIABLE	39.106	7	5.587	98.670	.000
	SET	.166	22	7.563E-03	.134	1.000
	ERROR	8.719	154	5.662E-02		
Machine	VARIABLE	41.352	7	5.907	153.205	.000
	SET	8.624E-02	24	3.593E-03	.093	1.000
	ERROR	6.478	168	3.856E-02		
Electronics	VARIABLE	269.574	7	38.511	941.661	.000
	SET	.789	136	5.800E-03	.142	1.000
	ERROR	38.933	952	4.090E-02		
Motor Co	VARIABLE	46.350	7	6.621	157.102	.000
	SET	.149	28	5.331E-03	.126	1.000
	ERROR	8.261	196	4.215E-02		
Clothing	VARIABLE	70.603	7	10.086	260.627	.000
	SET	.254	49	5.182E-03	.134	1.000
	ERROR	13.274	343	3.870E-02		
Wool	VARIABLE	31.560	7	4.509	112.780	.000
	SET	7.256E-02	19	3.819E-03	.096	1.000
	ERROR	5.317	133	3.998E-02		
Mi Textiles	VARIABLE	62.480	7	8.926	162.618	.000
	SET	.179	35	5.106E-03	.093	1.000
	ERROR	13.448	245	5.489E-02		
Leather	VARIABLE	24.710	7	3.530	84.900	.000
	SET	5.146E-02	15	3.430E-03	.083	1.000
	ERROR	4.366	105	4.158E-02		
Food	VARIABLE	244.259	7	34.894	568.014	.000
	SET	.507	110	4.608E-03	.075	1.000
	ERROR	47.302	770	6.143E-02		

Table 5: Tests of between-subjects Effects for 1987, mixed model where SET is the random effect (size) and VARIABLE is the fixed effect.

to other variables. In terms of specific (as opposed to common) variability, items such as Sales, Wages or Current Assets mostly reflect size. Inside each industry their specific variability is small. Inventory or Funds Flow have more variability of their own. Finally, Fixed Assets, Working Capital and especially Long Term Debt, have a large proportion of variability of their own. Correlations amongst clusters of variables create even greater limitations. Liquid assets and liabilities, for instance, are correlated amongst themselves and with Sales. The number of employees is correlated with Wages (large firms often pay higher salaries than small firms).

Given the above, it is clear that a good size estimate would require more items than those available in this study. Even so, using the model

$$s = \frac{1}{7}\left[S + NW + W + D + CL + I + N\right] \tag{8}$$

a significant decay in the final cross section variance of s_J, between 6% and 20% when compared with that of Sales, is obtained for most of the industries and years. It is understood that (8) is a schematic representation of the true model where the symbol corresponding to each item represents a mean-adjusted logarithm, $\log x_k - m_k$ of the corresponding variable from which thresholds, when significant, were removed.

Figure 1 illustrates, for the major industrial groups anf for the period 1984–1987, how the cross section variance of size (expressed as a percentage of the variance of Sales) is affected by the entry into (8) of each new variable component. A general tendency towards a decreasing variance is apparent, in spite of exceptions such as the case of the Food industry where, for most of the years considered, the entry of some new variables into the model actually increases the variance of the estimated size.

The Distribution of Size

(THIS SECTION IS UNDERGOING SOME CHANGES) It was mentioned above that, in cross section, distributions of logarithms of accounting variables are symmetrical but slightly leptokurtic. Specifically, no cases of negative kurtosis are observed for any considered variable and year. The same is observed in the logarithms of ratios formed from these variables with the difference that, in ratios, leptokurtosis is higher than in each of their individual variable components.

The distribution of s_J is different from that of logarithms of individual accounting variables in that leptokurtosis is subdue. Given the method used for estimating this parameter it is natural that the resulting distribution is closer to normality.[8] However,

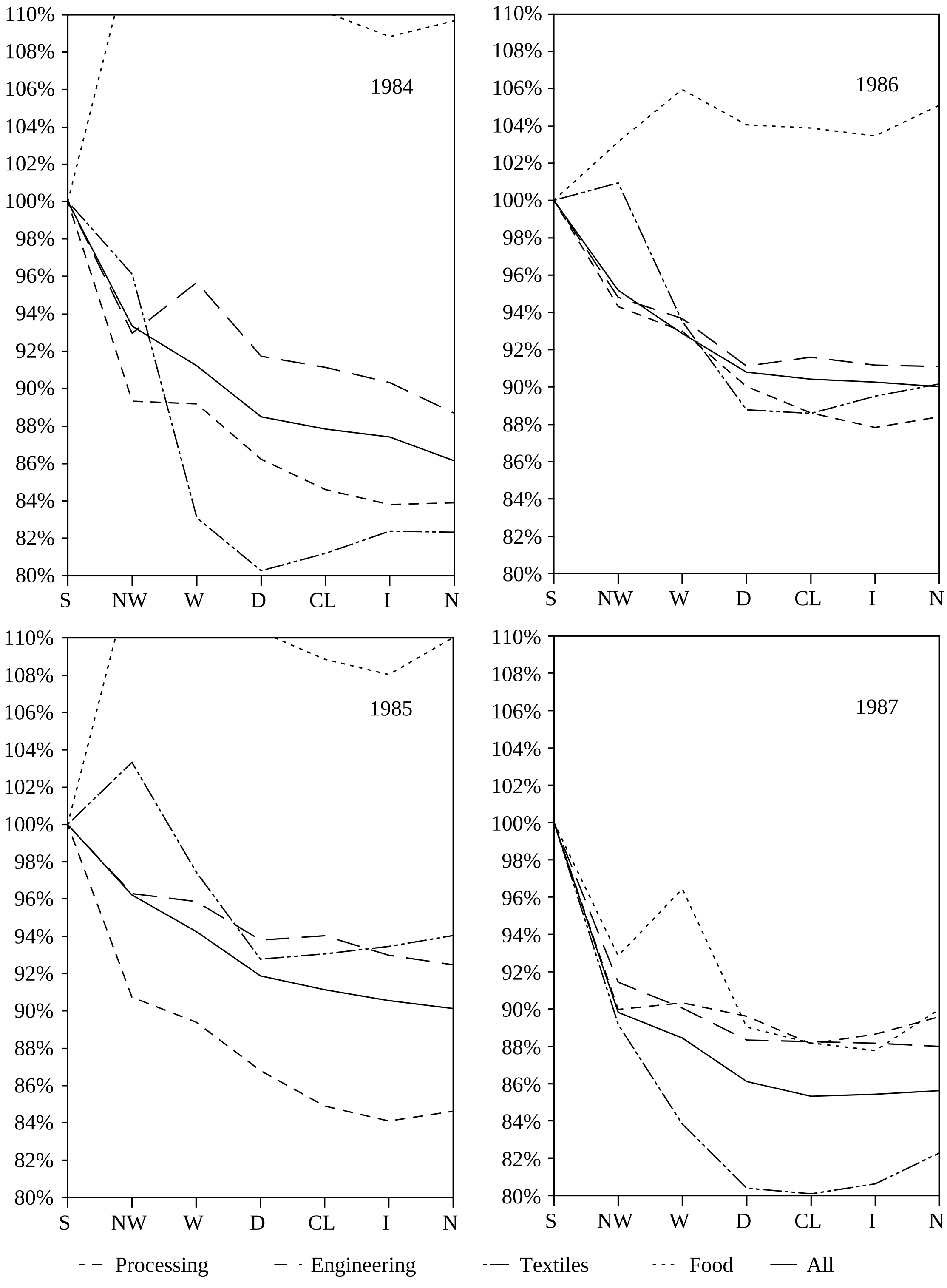

Figure 1: Percentage decrease in the variance of the size estimate when a new variable is included into the model.

Year		S	+NW	+W	+D	+CL	+I	+N
1983	Mean	0	8.0E-5	4.6E-3	4.2E-3	3.4E-3	4.1E-3	4.9E-3
	Std.Dev.	.676	.669	.666	.658	.657	.655	.655
	Skewness	-.037	.210	.324	.293	.317	.325	.360
	Kurtosis	1.108	.536	.484	.498	.472	.476	.481
1984	Mean	0	-5.5E-4	5.1E-3	5.9E-3	8.4E-3	8.7E-3	1.1E-2
	Std.Dev.	.717	.702	.700	.687	.685	.683	.680
	Skewness	-.078	.250	.342	.312	.318	.311	.369
	Kurtosis	.888	.276	.310	.339	.350	.387	.378
1985	Mean	0	3.8E-4	5.8E-3	5.7E-3	8.2E-3	8.4E-3	1.1E-2
	Std.Dev.	.684	.681	.677	.666	.665	.662	.661
	Skewness	.098	.337	.452	.399	.415	.409	.461
	Kurtosis	.795	.315	.311	.382	.387	.420	.425
1986	Mean	0	3.0E-3	9.4E-3	9.6E-3	1.1E-2	1.1E-2	1.2E-2
	Std.Dev.	.682	.672	.668	.658	.658	.657	.658
	Skewness	.042	.284	.406	.367	.386	.388	.414
	Kurtosis	.671	.431	.387	.456	.457	.468	.489
1987	Mean	0	4.9E-3	9.9E-3	1.2E-2	1.2E-2	1.1E-2	1.1E-2
	Std.Dev.	.693	.662	.660	.649	.647	.647	.650
	Skewness	-.196	.226	.320	.317	.366	.369	.391
	Kurtosis	1.336	.660	.603	.588	.557	.536	.549

Table 6: Evolution of the mean, standard deviation, skewness and kurtosis when variables are added to the model for estimating size. All industries together.

there most exist another, more powerful explanation for the decline in leptokurtosis since it seems to occur wherever the logarithms of two accounting variables are added, not just in the case of averages of many such variables. Table 6 shows an example of this. When, e.g., the logarithm of Net Worth is added to the logarithm of Sales, the kurtosis of the resulting variable decreases significantly.

The reason for this may be straightforward. The variability of s_J spreads along the main axis of a multivariate distribution. Such distribution is the source of the Gaussian behaviour of logarithmic variables.

Figure 2 is a *contour plot* depicting the density (or bivariate distribution) of accounting variables $\log Y$ and $\log X$. Logarithmic size is measured along the 'Size Axis'. The source of positive kurtosis is the 'Ratio Axis'. $\log Y$ and $\log X$ are 45^o projections of this bivariate distribution. In logarithmic space the deviations from the median of the ratio Y/X are a difference of two residuals ... Their distribution is the result of the subtraction of two Normal distributions with the same mean. But these two distributions are very similar in spread. A large fraction of the variability of items comes from the strong, common, effect they share. The remaining variability is the source of the positive kurtosis in log space. It is so small a proportion of the total one that it could have any distribution without greatly affecting the overall lognormality of items. In ratios, however, it is prevalent. Such a density is high in one of its main dimensions and small in the other one. The largest dimension accounts for most of the variability.

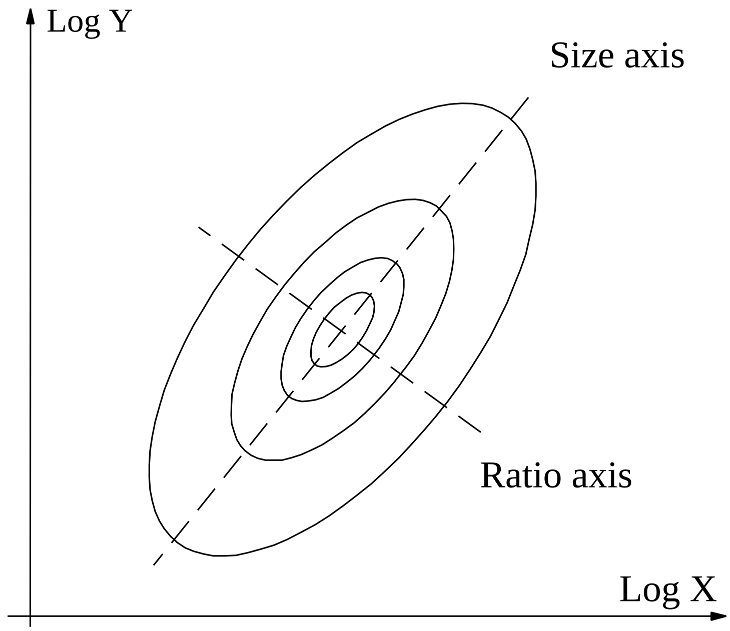

Figure 2: Bivariate distribution of two accounting variables in logarithmic space.

The variability of ratios is explained by the smallest dimension, the ratio axis. It is orthogonal to the size one, which accounts for the variability introduced by size.

When an observer positions himself so that the largest dimension of this surface becomes parallel to his horizon, he sees a Gaussian shape. When he observes it transversally, it yields a leptokurtic shape. Thus, the weak, particular, effect is the source of leptokurtosis in accounting data, and the strong, common, one is the source of their lognormality. For example, the small amount of kurtosis observed in the logs of raw numbers denotes the influence of their own variability superimposed to a common, Gaussian, one.

Discussion and Concluding Remarks

The paper has defined firm size as a random effect, the same for each set of accounts. Empirical evidence suggests that such effect explains a huge proportion of the variability of accounting numbers. Of course, the fact that the variability unexplained by the model be very small should not lead to the conclusion that it is non-important.

This definition stems from the assumption that accounting variables exhibit a multiplicative rather than an additive behaviour. Although the reasoning leading to the definition of size further assumes proportionality amongst accounting variables, later it is shown that this second assumption can be relaxed, namely in the case of constant terms. It may be stated, therefore, that the premises leading to a definition of firm size are neither too restrictive nor alien to existing empirical evidence. Indeed, the multiplicative character of accounting numbers is an established fact whereas deviations from strict proportionality are, in general, small and well understood (Sudarsanam and Taffler, 1995).

Discrepancies in relation to the basic assumption of the paper are likely to occur as follows: first, although a strictly lognormal behaviour is quite typical of samples of listed companies such as those in the EXTEL tape, when smaller firms are also included the skewness of distributions increases; second, samples drawn from financially distressed firms also lose their proportionate character; finally, as Whittington (1980) has pointed out, besides constant terms there are other causes leading to non-proportionality amongst accounting variables. Non-linearities caused by economies of scale, for instance, would bias the proposed estimate of firm size.

According to the paper, size may be estimated using two equivalent forms. First, size may be viewed as the number of times a given set of accounts is larger or smaller than the unit. This description of size is intuitive but its multiplicative character precludes a direct use in parametric models. The second possible form, logarithmic size, is adequate for statistical manipulation but its interpretation (a continuously compounded rate of change) may be less intuitive. It may be useful to view logarithmic size as the rate of growth capable of transforming, during a period of one year, a firm of a pre-specified (unit) size into a firm of the present magnitude.

It may seem surprising that the widely explored condition of proportionality ($Y/X =$ Constant) may yet be able to apportion a higher degree of definition to the concept of firm size. This is probably because scale invariance, being a structural form, naturally uncovers the mechanism underlying proportionality. Incidentally, scale invariance should be regarded, not just as a way of extracting interesting consequences from the hypothetical validity of the ratio method, but also as a means of providing theoretical support to real-world findings and, in general, as a way of better understanding the statistical behaviour of accounting numbers. In fact, equality (3) closely models two essential characteristics that are present during the generation of accounting variables. First, since these variables are taken from the same set of accounts, they are generated under the same size influence as indicated by (3). Surely, it is natural to assume that a collection of numbers taken from the same firm and period should contain a common source of variability. Indeed, what is difficult to accept is the assumption that numbers from the same report have nothing in common, nothing capable of making them more similar than those in other reports. Second, the generation of accounting numbers is a process of accumulation, thus obeying a multiplicative law of probabilities consistent with the relative nature of common changes in (1) or (3). Accounting variables (like other economic variables such as income, wealth, stock prices and the size of firms) cannot be described as resulting from the kind of additive stochastic process which underlies normal variables. Whilst each transaction contributing to the amount reported as, say, Total Sales for a given period is itself a random event, an individual transaction contributes to the reported aggregate not in a manner which could lead to either an increase or decrease in Total Sales, but by accumulation only. Accumulations of random events tend to be multiplicative, as opposed to additive, because the likelihood of realisations is conditional on the occurrence of a chain of several previous events. Such likelihood thus stems from multiplying, rather than adding probabilities.

It may be argued that the proposed definition of firm size is not *sufficient* as it is not established beyond doubt that it captures all the important characteristics of size. It should be noticed, however, that the proposed definition captures the most important and intuitive characteristic of size, viz, relative magnitude. Moreover, since other possible characteristics have nowhere been defined and since there is no theoretical basis to do so, size will have to be presented in the form of an axiom and judged by its conformity with our present knowledge of how firms grow.

The proposed definition is the simplest formulation consistent with such knowledge. In other words, there may exist other definitions of firm size but the proposed definition is the simplest one. Thus common sense and proper scientific method recommend that those more elaborated definitions should not be considered unless in the presence of shortcommings of the simplest definition.

All in all, the premise that relative differences observed in accounting numbers reflect, *inter alia*, differences in the size of those firms (or, in a time series context, the growth rate of the firm), should be viewed as natural and easy to accept. Moreover, both empirical observation and common sense advocate in favour of the existence of a well-defined size. Not only is it an empirical fact that the financial statements of large firms contain reported numbers which are many orders of magnitude larger than those reported in the accounts of small firms, there are also compelling economic reasons to support the conviction that each firm's actual size greatly influences the overall magnitude of numbers reported in its accounts. Indeed, if variables such as Sales or Earnings were not closely related to size, then profitability and dividend yield would be diluted by any increase in size and firms would carefully avoid growing.

The difficulties felt when trying to select variables to build a model of size stem from the limited number of items available and from their relatively high level of aggregation. It is expected that these difficulties will be avoided by the use of comprehensive data sets since a larger number of available variables increases the feasibility of the model.

The major contribution of the paper probably is to add to our understanding of the statistical characteristics of accounting numbers. The doubt as to whether it is possible to give a precise theoretical definition or statistical description of size should give way to the knowledge that size actually is the dominant source of variability of accounting numbers. As for the practical consequences of the paper, maybe the most evident is the possibility of building deflators able to closely approach size. Simple ratios such as

$$\frac{\text{Deflated Variable}}{\sqrt{S \times TA}} \quad \text{or} \quad \frac{(\text{Deflated Variable})^3}{S \times TA \times N}$$

should, according to the developments presented above, increase the content in size of the deflator, thus reducing the undesired removal of interesting effects in the deflated variable. More elaborate techniques will allow the estimation of almost pure size. This, in turn, will lead to a better understanding of the behaviour of variables such as Sales, Working Capital or Debt and also to the finding of clusters of variables exhibiting significant after-size correlation.

Most of the situations where size is proxied by variables such as Sales or Market Capitalisation should continue to be so as the reasons for using these proxies go beyond the simple imitation of size. As Lev and Sunder (1979, p. 195) recall, it makes sense to 'measure a firm's size by Sales in a products market, by employees in a labour market, by materials in a material market, by assets in a capital market'. However, this multiple use of proxies for size is purely functional and should not be taken as an indication that size may be multi–dimensional.

When, for instance, the Companies Act, in defining small companies, presents three different measures of size, it should be understood that either each of these measures has its own functional area of application or the hitherto poorly defined concept of size did not allow a sharper definition. What should not be concluded is that companies might be capable of growing in two or three orthogonal directions. Indeed, the concept of a multi–dimensional size is so alien to reality, so full of contradictory consequences, that the *onus* of proving its existence should rest with those who have mentioned it in the first place.

Notes

1. See Lev and Sunder (1979, pp. 194–198) for a discussion of other difficulties related to the proxy for firm size.

2. Well-known deviations from the Gibrat's Law (small firms seem to grow faster than large firms, for instance) are not deemed as sufficiently important to allow questioning the proportionate character of size.

3. Furthermore, where items can take on positive and negative values, then lognormality is observed in the subset of positive values and also in the absolute values of the negative subset. The same author has also uncovered cases of three-parametric lognormality. While the Normal distribution is completely specified by the mean and standard deviation, the lognormal distribution may require one extra parameter in order to account for overall displacement of the distribution. Where a displacement of item x (say $x - \delta$), and not x itself, is Normal after a logarithmic transformation, the distribution of x is known as *Three-Parametric* Lognormal. The range of x is thus $\delta < x < \infty$. The usual, *Two-Parametric*, lognormal distribution is a special case for which $\delta = 0$. Since δ is a lower bound for x, it is known as the *threshold* of the distribution (Aitchison and Brown, 1957).

4. Three aspects of scale invariance are worth recalling.

   First, (1) describes a generative process, not any observed value. So long as the mechanism generating Y and X predicts that dY/Y is similar to dX/X, proportionality is verified irrespective of observed relative differences being similar or not. Indeed, without pre-supposing scale invariance, the analysis of firms using the ratio method is impossible. For example, unless it can be *a priori* assumed that structural relative differences across firms of any size are the same for Current Assets and Current Liabilities, it will not be possible to accept whether a Current Ratio above the norm in a large firm means liquidity above the norm or rather a structural difference whereby large firms exhibit a pattern of liquidity which is different from that of medium- and small-sized firms.

   Second, proportionality is not necessarily affected if other statistical effects, not just scale invariance, are present during the generation of accounting numbers. The requirement is that the effect of scale invariance is independent from other influences.

   Finally, as explained by Trigueiros (1997) it is possible to correct for the existence of additive terms in an otherwise proportional relationship. Thresholds δ_Y and δ_X of the distribution of Y and X respectively may be accounted for by using

$$\frac{dY}{Y - \delta_Y} = \frac{dX}{X - \delta_X}$$

rather than (1).

5. Components of the unit set must also be proportional, i.e., for any possible pair $k \leq N, l \leq N$ it should be verified that

$$\mathcal{X}_k/\mathcal{X}_l = \text{Constant}_{k,l}$$

where all $\text{Constant}_{k,l}$ are similar to those in (2). As a consequence, only one of the $\mathcal{X}_k$ values may be chosen arbitrarily, all the remaining $\mathcal{X}_k$ ensuing from such initial choice.

6. Aitchison and Brown (1957), Ijiri and Simon (1977) and other authors describe in detail the generative mechanism and the assumptions leading to the lognormal or to other multiplicative distributions. According to (3), the natural form governing the behaviour of accounting variables $x_1, \cdots, x_k, \cdots, x_N$ may be described as

$$\frac{dx_k}{x_k} = r(\tau)d\tau. \tag{9}$$

where τ is the variable in terms of which the x_k are to be described. Sales, for instance, may be described in terms of unit price, economic or other influences or simply as evolving over time. Where a cross-section description is desired, τ is supposed to measure the *location* of each firm's set of accounts along the range of all possible values of the variable. Notice that relative differences amongst sets of accounts should be described in terms of changes $d\tau$ in τ since they are now viewed as a function of τ.

In relation to the above-mentioned research, the assumption that should be added in this specific case is that the relative change $r(\tau)$ is the same for all accounting variables so that scale invariance is verified. If it is further assumed that r is independent of τ, i.e., $r(\tau) = r$ then $x_1, \cdots x_k, \cdots, x_N$ will obey the Gibrat's Law, exhibiting a proportionate behaviour. A simplified cross section solution of (9) is

$$x_k(\tau) = \mathcal{X}_k \exp(r\tau)$$

Therefore, s_J, the logarithmic size for set J, is equal to $r\tau$ and (6) may also be written in terms of the continuous variable τ rather than in terms of subscript k:

$$\log x_k(\tau) = \mu_k + r\tau + \varepsilon_{kJ}. \tag{10}$$

The meaning of r is that of a *logarithmic standard deviation* of size. Industries where firms range from the very small to the very large exhibit large r whereas those where size is more regular exhibit small r. In turn, τ measures standardised size, i.e., the number of logarithmic standard deviations separating the position of a specific set of accounts along x_k from the median, $\mathcal{X}$.

7. Statistical effects capture differences in relation to the expectation, where such differences are introduced by components of the variance. Non-random influences are referred to as 'fixed effects' because the component introduced by their levels is deterministic. Where the component is itself a random variable, as in the present case, the effects are called 'random effects' and forms where both types of effects are present are known as 'mixed models'. See, e.g., Snedecor and Cochran, 1965, pp. 237–240.

8. An addition or averaging of distributions tends to normality.

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