# A Re-examination of the Theoretical Foundations of Ratio Analysis

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## Contents



**Abstract**

Accounting academics have often considered the widespread use of financial ratios as somehow intriguing. The paper investigates the consequences of assuming that ratios can be validly used by analysts, comparing the ensuing conclusions with those of other authors. In the light of this methodology, some major objections in relation to the ratio method emerge as less relevant or inadequate. This, in turn, suggests that users of ratios may be acting more rationally than previously thought.

# 1 Introduction

The academic community has remained largely sceptical about the validity of financial ratios, contending that their use stems from routine rather than from rational consideration. Indeed, in an early contribution to this topic, Horrigan (1965) remarked that financial ratios were referred to in text books in almost apologetic tones as though their expected utility were extremely low. Horrigan's response to this situation, however, was to seek to dissipate such doubts by describing the statistical characteristics of some widely used ratios in order to demonstrate that they may be useful. Following Horrigan's optimistic review, subsequent empirical research revealed some promising applications of financial ratio analysis. Beaver (1966) and Altman (1968), for example, showed that ratios have the potential to predict bankruptcy.

A few years later, however, the scepticism returned, with authors such as Deakin (1976) noticing that the empirical frequency distributions of ratios appear to vary widely and, as a result, questioning the validity of analytical methods which assumed the normality of ratio data. This prompted Frecka & Hopwood (1983) and others to propose *ad hoc* techniques such as the trimming of outliers to deal with the unduly influential values present in samples of financial ratios, techniques which reflected the apparently widespread belief that there were no general rules underpinning the ratio method.

Adding to the growing doubt concerning the validity of financial ratio analysis, Lev & Sunder (1979) raised some fundamental questions as to whether the use of ratios is motivated by well-founded considerations or whether, in contrast, it is merely a tradition. These authors concluded that almost all of the assumptions required for valid ratio analysis are likely to be violated in practice. Whittington (1980) also uncovered cases where ratios seem not to be up to the task but distinguished the normative application of ratios from their less-acceptable use in prediction.

Both Lev & Sunder (1979) and Whittington (1980) stressed that valid measurement using ratios requires proportionality between the components (i.e., $Y = bX$). Since such an assumption seems to be too restrictive, these authors advocated a two parameter regression model ($Y = a + bX$), or similar functional form, rather than the single parameter ratio model. Barnes (1982) went further, suggesting that non-proportionality is also in the origin of the skewness often found in the distributions of ratios. The prevailing scepticism about the usefulness of financial ratios deepened further when Tippett (1990) claimed that ratios such as those used as standards to perform comparisons (industry norms, benchmarks and others) were intrinsically unstable, drifting upwards or downwards over time.

Rather than seeking the outright condemnation of the ratio method, other authors have tried to understand why financial ratios are so widely used. McLeay (1986*a*), for instance, has demonstrated that a fuller understanding of ratios can be achieved by taking account of the behaviour of the variables from which financial ratios are constructed, particularly where firm size plays an important part. At the same time, some limiting case theoretical ratio models which allow for exponential growth in accounting variables were identified (McLeay, 1986*b*). Trigueiros (1995) offered a simple explanation for the diversity of statistical distributions found in ratios and pointed out that the multiplicative[1] character of accounting numbers suggests the existence of a statistical effect which is common to all of the figures reported in a particular set of accounts but which varies both through time and across firms. This was further expanded in Trigueiros (1997) where such effect was presented as the manifestation of the pervasive influence of firm size on the magnitudes of accounting numbers and where it was also shown that non-proportionality may not necessarily be damaging for ratio analysis.

This paper investigates the premise that the ratio method is valid, comparing the ensuing conclusions with those of other authors. Such comparison suggests that the ratio method is basically sound[2] and that the published objections in relation to the use of ratios may be attributed to unduly restrictive assumptions or to poor attention to the context where ratios are used. The paper also shows that firm size is a well-defined variable implicit in the concept of proportionality and that the multiplicative character of accounting variables is not just supported by empirical observation but is a prerequisite of valid ratio usage. The supposed drift found by Tippett (1990) in ratios is also discussed. Finally, the paper proposes new statistical models arguably capable of better portraying the behaviour of accounting numbers together with a new interpretation of accounting differences and negative values.

## 2 The Two Postulates of Ratio Analysis

Ratios present one number as a representation of another, scaled up or down by a given factor. The Interest Cover ratio, for instance, is intended to show the number of times Earnings is greater than Interest; the Sales Margin ratio aims at expressing Earnings, the numerator, as a fraction of Sales, the denominator and so on.

Financial analysis is just one of the many tasks where scaling is used. Maps and other models, for instance, are also governed by a ratio showing the number of times any measurement in the representation is smaller than in reality. It is not surprising, therefore, that the first and most basic postulate of ratio analysis, the same as that governing maps, simply states that the scaling factor should remain constant. In maps, a constant scaling factor is applied to all measurements so that small objects are depicted as proportionally small representations and large objects are depicted as proportionally large representations. In ratio analysis, ratios are supposed to have the same financial meaning no matter the size of the firm being analysed.

It should be recognised, however, that the requirements of financial analysis stretch beyond those of scaling. First, in ratio analysis, the scaling factor is not a convention as in maps. Rather, it is supposed to reflect some pre-existing reality or standard: the natural relationship between two accounting variables (a norm), a benchmark, a goal to attain, a prediction[3] or a ratio taken from a previous period. Second, in ratio analysis, any *discrepancy* that may be observed in relation to this pre-existing standard is considered as valuable information[4] whereas in maps it is discarded as an error.

What makes financial ratios potentially *invalid* is this comparison that is made with a pre-existing standard.[5] If, for instance, the natural relationship between the numerator and the denominator of a ratio, although pre-supposed to be a simple scale, actually cannot be so, then such ratio is indeed invalid since the information it conveys is misleading. The traditional approach has been to question the validity of financial ratios on these grounds, i.e., that a natural scale between the numerator and the denominator of the ratio may not exist. Unfortunately, rather than investigating the existence or not of natural scales, the extant literature has adopted a more stringent stance, that of searching for *expected* ratios that are also the natural scale between the numerator and the denominator. This extra requirement, unduly restrictive, was implicitly taken in when authors inadequately attempted to describe financial ratios using statistical models of the additive type. It led to a great deal of pessimism about the validity of ratios because no plausible random mechanism has been found such that its expected ratio is independent from the magnitudes of the components of that ratio.[6]

Another negative consequence of this assumption was that an important postulate of valid ratio usage remained concealed as authors improperly incorporated it into such additive assumption. This postulate requires that observed discrepancies in relation to the standard should be independent of the size of the firm. Specifically, for a given ratio, the likelihood of observing a discrepancy of, say, 1%, in relation to the norm, must be the same for small and large firms. Should this not be the case, then the relevance of a 1% over-performance or under-performance would vary, depending on the size of the firm. This too would render ratios useless. Another basic postulate of ratio analysis is therefore homeocedasticity, that is, the distribution of the measurement must be independent of size.

Lev & Sunder (1978) have discussed analytical models of ratios where this postulate is violated. In one such instance they comment that 'since the deviations [from the norm] are [expected to be] small for large firms and large for small firms [thence] the formation of ratios and their comparison to other [...] ratios does not provide an adequate means of control for size' (p. 191). Not just Lev & Sunder but also other authors have came across instances of this postulate. Invariably, they have labelled them as cases of model mis-specification rather than as potential sources of invalidity of ratios in their own right. Such labelling, however, is based on the premise that only expected values should be used as ratio standards and norms. Where a standard is not an expected ratio, or when it has a financial rather than statistical meaning (benchmarks, predictions), the homeoscedastic condition cannot be taken as the consequence of a model's correct specification. Therefore, two postulates of ratio validity, not one, should be investigated separately.

In order to avoid unwanted assumptions, this section first derives the *natural form* of proportionality. Natural forms describe general mechanisms or structures rather than observed behaviour. They are often expressed as differential equations and the paper uses uppercase letters in variables to signal their use.

### 2.1 First Postulate: Proportionality and Firm Size

A contradiction often found in papers on the validity of ratios is that, while no proper definition of size is offered, conclusions invariably stress that ratios must fail to remove the effect of size from the financial measurement. If authors do not define size, how can they assert that ratios fail to remove it? This section aims at finding out what ratios actually are capable of removing. After all, this is the question that *must* be answered at the start of any argument about the usefulness of ratios. It is shown here that the first postulate of ratio analysis, proportionality, implies the existence of a well-defined variable with the attributes of size and that this new variable is plausible i.e., it is natural to suppose its existence in the case of accounting variables.

It was mentioned that the most basic condition for the valid use of financial ratios is proportionality between the numerator of the ratio, $Y$, and the denominator, $X$, so that $Y/X$, the scaling factor, is constant (Whittington, 1980). Authors typically illustrate the concept of proportionality by referring to its two pre-requisites, namely linearity and the absence of an 'intercept term'[7] in the relationship between $Y$ and $X$. These pre-requisites, however, emphasise the relationship between $Y$ and $X$ but they are less effective in showing how changes in $Y$ (across firms or in the same firm for different periods) relate to the corresponding changes in $X$. Yet, a thorough understanding of ratios requires a dynamic approach where changes are also included in the analysis. This can be achieved by using the differential equivalent to $Y/X$ = Constant. When $Y$ is proportional to $X$, the rate of change of $Y$ with respect to $X$ is also constant and similar to the ratio itself. For ratios to be valid, therefore, the following must hold:

$$\frac{Y}{X} = \frac{dY}{dX}$$

where $dY, dX$ are any related changes or differences observed in $Y$ and $X$.[8] This formulation fully encompasses the earlier definition, having the advantage of highlighting potentially interesting facets of proportionality. For instance, by re-arranging terms, the above becomes

$$\frac{dY}{Y} = \frac{dX}{X}. \tag{1}$$

Equality (1) shows that the implicit condition of proportionality is *scale invariance*, whereby the relative changes in $Y$ are equal to the relative changes in $X$. For instance, when comparing firms in cross-section, if the Current Assets figure is many times larger in one firm than in another, then this should also be the case for Current Liabilities. Similarly, in a time-series analysis, (1) implies that variables eligible as components of a ratio should grow at the same rate. If, say, Sales grows by 12% during a given year then Earnings should also grow by 12% during that year.

Remember that (1) describes a mechanism, not realisations. So long as a mechanism underlying reported numbers predicts that $dY/Y$ is similar to $dX/X$, proportionality is verified irrespective of the observed differences between $Y$ and $X$. As already stressed, without pre-supposing scale invariance it would be impossible, using ratio analysis, to arrive to any reliable conclusion about the financial characteristics of firms. For example, unless it is postulated that relative changes across firms of any sizes are the same for Current Assets and Current Liabilities, it would not be possible to infer whether a Current Ratio above the norm in a large firm is attributable to the liquidity of that particular firm or is just a characteristic of large firms.

Scale invariance importantly suggests that numbers taken from the same set of accounts are influenced by the same effect. In fact, for each valid ratio, two of the items reported in a set of accounts (the numerator and the denominator of that ratio) are structurally linked by (1), sharing the same source of variability. Since the validity of the ratio method rests on the validity of several, widely used ratios, not just on one or two isolated cases, and given that items used to form one of such ratios are often used to form other ratios as well, it follows that there must exist a common source of variability underlying several of the items found in published sets of accounts.

Sales, for instance, is used by analysts to form Margin, Turnover and Profitability ratios together with variables such as Total Assets, Cost of Goods Sold or Gross Profits. Equality (1) thus requires that relative differences or changes observed in those variables are the same as in Sales. Total Assets is used to form Leverage ratios thus extending this requirement to Net Worth or Long term Debt. Since the subtraction of two variables preserves common relative changes, Total Liabilities (Total Assets minus Net Worth) also has to obey the same requirement. Cost of Goods Sold, in turn, extends it to Inventory via the popular Inventory Turnover ratio; Net Worth, being the denominator of Return on Equity ratios, extends it to Net Income, thence to Dividends and Interest Expense. Indeed, it may be difficult to identify aggregate items from accounting reports that analysts never combine with other items to form several, popular ratios, or that are not forced by accounting identities to change at the same rate as other items.[9]

In short, if all useful ratios, formed with $N$ different items $X_1, \cdots X_N$, are postulated to be proportional, then

$$\frac{dX_1}{X_1} = \frac{dX_2}{X_2} = \cdots = \frac{dX_N}{X_N}. \tag{2}$$

This extended condition of scale invariance is indeed a consequence of the assumption that the ratio method as a whole is valid. It clearly suggests, as mentioned, the existence of a common source of variability in numbers from the same set of accounts.

It is now easy to show that such common source of variability possesses the attribute of a size measurement. Consider two sets of accounts, $b$ and $a$, issued by the same firm in different periods (the time series context) or by two different firms in the same year (the cross section context). $b$ and $a$ may be viewed as two realisations of the same $N$ variables $X_1, \cdots X_N$ and, in (2), differences between them are written

$$\begin{aligned} dX_1 &= X_{1b} - X_{1a} \\ &\vdots \\ dX_N &= X_{Nb} - X_{Na} \end{aligned}$$

thus, using (2), it is possible to express items in the set of accounts $b$ in terms of items in the set of accounts $a$ as

$$\begin{aligned} X_{1b} &= X_{1a}(1 + r_{ba}) \\ &\vdots \\ X_{Nb} &= X_{Na}(1 + r_{ba}) \end{aligned}$$

where $r_{ba}$ is the common relative difference by which an item in $b$ differs from the corresponding item in $a$. Therefore, structurally at least, if one item is larger in $b$ than in $a$, it follows that *any other item* will also be larger in $b$ than in $a$ and conversely. If, say, Current Assets is larger in $b$ than the corresponding Current Assets in $a$, then Sales as in $b$ will also be larger than the Sales figure reported in $a$ and so on. It is thus possible to say without ambiguity that the set of accounts $b$ is, as a whole, larger or smaller than $a$. This possibility of comparing the size of two sets of accounts makes it possible to rank by size, without ambiguities other than those created by identical sets, not two but any collection $a, b, \ldots, j, \ldots$ of sets of accounts.

The existence of a size measurement underlying the postulate of proportionality is thus demonstrated. If ratios are valid it should be possible (by applying statistical sampling and modelling techniques) to estimate the relative sizes of firms and the corresponding confidence intervals. For instance, a continuous size measurement may be constructed by postulating a suitable *unit size* set, $\mathcal{X}_1, \cdots \mathcal{X}_N$ and then measuring the size $S_j = 1 + r_j$ specific to the $j^{th}$ set of accounts against this unit. Such measurement would show how many times the firm publishing the $j^{th}$ set of accounts is larger or smaller than the unit.

In such case the formulation describing the $k^{th}$ item is

$$X_k = \mathcal{X}_k \, S_j \tag{3}$$

where the level $\mathcal{X}_k$ is the value of item $k$ in the unit set and $S_j = 1 + r_j$, the size of the $j^{th}$ set of accounts, is the number of times an item in $j$ is larger or smaller than its level. Ratios formed from two such items remove the common factor, $S_j$, while uncovering the number of times by which their components differ in the unit set. It may also be concluded that proportionality directly leads to the modelling of accounting variables as the *product* of a constant level by a size factor.

It was mentioned that authors often voiced reservations about the ability of ratios to fully remove size. The reasoning just presented suggests that, actually, it is the opposite problem (that of ratios removing more than just size) that is more likely to occur. Size is what all ratios remove by virtue of the common origin of components. An individual ratio, however, will also remove any multiplicative effect that is present in its numerator and denominator but not in all the other items of that set of accounts. Suppose, for instance, that Current Assets and Current Liabilities both reflect, not just the effect of size (caused by the fact that these items are taken from the same set of accounts), but also an effect specific to liquid funds only. In such case, the Current ratio is the result of removing, not just size, but also this liquid funds effect. One practical consequence of this is that ratios aiming specifically at removing size should be formed with care lest potentially interesting influences are lost.

It may be argued that proportionality is probably verified within specific clusters of items but not necessarily amongst them all. In other words, a set of accounts may possess two or three size dimensions, not just one. Notwithstanding the cases of recently formed firms, firms in financial distress or other less general cases, the hypothesis of size being multi-dimensional seems difficult to sustain. In fact, while financial statements of large firms contain reported numbers that are many orders of magnitude larger than those reported in the accounts of small firms, inside each set of accounts items do not differ by as much: one item that is, say, one million times larger or smaller than other items is an exception, not the rule. Yet, more than one size dimension would lead routinely, not in exceptional cases only, to numbers with six or more digits together with numbers with two or one digit. This would also render the principle of materiality as impossible to apply.

Should more than one size dimension exist, then a ratio formed with items belonging to non-agreeing dimensions would exhibit typical magnitudes similar to those of raw items or their reciprocals, i.e., billions as well as micros. Again, such a ratio may indeed occur but only as the result of conditions not applicable generally. The fact that, after many years of experimentation with ratios of all types, microscopic or very large ratios are considered as atypical, clearly shows that items from the same set of accounts share, in general, a unique size influence.

There are also compelling economic reasons to support the conviction that each firm's actual size greatly influences the overall magnitude of numbers reported in its accounts. If variables such as Earnings were not closely related to size, then profitability and dividend yield might be diluted by any increase in size and firms would carefully avoid growing.

Indeed, scale invariance should be regarded, not just as a way of highlighting what ratios are supposed to remove, but as a means of improving the theoretical foundations necessary to correctly model accounting data. In fact, equality (2) captures two essential characteristics of accounting variables. First, it cannot be denied that accounting numbers taken from the same set of accounts are generated under the same size influence as indicated by (2). Second, the generation of accounting numbers is a process of accumulation, thus obeying a multiplicative law of probabilities consistent with the proportionate nature of changes as described in (1) or in (2).

As an introduction to the following section it may be mentioned that scale invariance also suggests the need to consider, not just proportionality but another postulate of valid ratio analysis. In (3), proportionality simply ensues from $S_j$ being independent from the level factor $\mathcal{X}_k$. If (3) is verified, then any $S_j$, no matter which, will lead to ratios obeying the traditional postulate of ratio validity. Indeed, proportionality is verified even if $S_j$ evolves in such a way as to make ratio analysis meaningless. This is basically is why another postulate is required.

## 2.2 Second Postulate: Exponential Changes

It was mentioned that the existence of the scaling factor is not sufficient to bring about valid ratios. The measurement using ratios (i.e., the difference between the observed ratio and the standard) must also be size-independent. As a corollary, relative changes in components of ratios are required to be size-independent too.[10] Variables where relative changes are size-independent are said to obey the Gibrat's Law, the source of the family of multiplicative processes where relative, rather than absolute changes are homogeneous.

In order to illustrate the practical consequences of this postulate it is necessary to model, not just differences or changes, but the joint *evolution* of accounting variables under the influence of firm size. Since the label $j$, introduced in the previous section in relation to sets of accounts, is insufficient to characterise an evolution, a new variable, $\tau$, is now introduced. It is thus supposed that, in principle, ratio components are a *function* of $\tau$ and, from (2), the mechanism driving relative changes in the $k^{th}$ item may be expressed as

$$\frac{dX_k}{X_k} = r_j\, d\tau \tag{4}$$

where $r_j\, d\tau$ is the same for items from the $j^{th}$ set of accounts so that scale invariance is verified. In the analysis of time series $r_j$ is an instantaneous rate of growth of $X_k$ in instant $\tau$. In cross section, $\tau$ is simply a scaling parameter since $r_j$ may directly measure how far a set of accounts is from another set taken as the reference.

Mechanisms such as (4) may generate many types of variables. Where, for instance, $r_j$ decreases in proportion to $X_k$, that is, where $r_j = \beta_k / X_k$ with $\beta_k$ expressing the constant rate of change of $X_k$ with $\tau$, then the resulting variable evolves linearly with $\tau$. Formulations such as

$$x_k = \beta_k\, \tau$$

would be, in this case, a solution of (4). Although $x_k$ is scale invariant, the second postulate of valid ratio usage is violated as relative changes in the variable decrease with $\tau$ i.e., $r_j = 1/\tau$ and it is $\beta_k$, not $r_j$, that remains constant.

As mentioned, the two postulates of ratio analysis are verified where $r_j\, d\tau$ is independent of $\tau$. In this case any $x_k$ evolving exponentially with $\tau$, e.g.,

$$x_k = \mathcal{X}_k[1 + r_j]^\tau \text{ for discrete } \tau \text{ or } x_k = \mathcal{X}_k \exp[r_j \tau] \text{ for continuous } \tau \tag{5}$$

is a solution of (4). The level $\mathcal{X}_k$ is the value of $x_k$ for $\tau = 0$.

Notice that (5) is a version of (3) as $\mathcal{X}_k$ relates solely to the item considered whereas the size multiplier (which, in this example is either $[1+r_j]^\tau$ or $\exp[r_j\tau]$) relates solely to the set of accounts considered. In contrast to (3), however, firm size is no longer undefined. Rather, it evolves in a specific, exponential way.

Where ratios are valid, therefore, the magnitudes of accounting numbers are described in a way that is similar to amounts accruing interest $r$ over a period $\tau$. Numbers taken from the same set of accounts, being under the same size influence, are portrayed as different magnitudes accruing the same 'rate of interest' over the same period. Across sets of accounts belonging to large and small firms, numbers are portrayed as similar magnitudes that have earned high or low 'rates of interest' respectively.

Examples of the two different contexts of ratio analysis (i.e., in time series analysis and in inter-firm comparison) are now given. In time series analysis, the interpretation of (5) is straightforward, $\tau$ measuring a sequence starting when the components of ratio $y/x$ assume the value of the corresponding item in the unit set, i.e., $y = \mathcal{Y}, x = \mathcal{X}$. Growth rates observed in $y$ and $x$, $r_y, r_x$ respectively, should be viewed as realisations of the same underlying growth rate $r$ so that scale invariance is verified.

Suppose that, in a given year, the reported Sales figure is £1,000 and Earnings is £100, then profitability is 10%. If Sales grows by $r$ (the same as the firm) but Earnings grows by only $r - n$, then, after a period of length $\tau$,

$$\text{Earnings} = 100 \exp[(r-n)\tau] = 100 \exp[r\tau] \exp[-n\tau]$$

and

$$\text{Sales} = 1,000 \exp[r\tau]$$

and the ratio decreases by $\exp[-n\tau]$. Not only is the ratio insensitive to the magnitudes of Sales or Earnings in the previous period, its decrease is also insensitive to the growth rate $r$. No matter small or large the growth of the firm may be, the change in Sales Margin is the same. The two postulates of the ratio method are thus verified. Moreover, if the disturbance $-n$ occurs during more than one period, the ratio will decrease by $\exp[-n]$ after one year, by $\exp[-2n]$ after two years and so on. That is, the disturbance accrues over time and it is possible to compare ratios relating to periods of different lengths just by accounting for such differences.

Other types of exponentiation, namely discrete, would produce similar results. By contrast, where components of ratios are governed by non-exponential processes, proportionality is verified but the measurement is size-dependent and disturbances do not accrue over time. For example, in a linear process, the effect of a disturbance $w$ in Earnings

$$\text{Earnings} = 100\, \beta_E \tau w$$

while

$$\text{Sales} = 1,000\,\beta_S\tau,$$

would make the measurement of profitability dependent on $\beta_S$ and $\beta_E$, the rates of growth of Sales and Earnings. Moreover, this effect would be independent of the time elapsed, a decline in Earnings of, say, 1% per year leading to a reduction of 1% in the ratio, regardless of the length of the period. For linear albeit additive disturbances, a reduction of $-n$ in Earnings in a given year,

$$\text{Earnings} = 100\,\beta_E\tau - n$$

while

$$\text{Sales} = 1,000\,\beta_S\tau$$

would lead to a reduction $n/(\tau\beta_S)$ in the ratio. Any slow down in Sales would amplify Sales Margin whereas any acceleration would subdue it.

In cross-section analysis, an intuitive meaning may also be given to $r$ and $\tau$. In this case $\tau$ measures the dispersion of size.[11] Industries where firms range from the very small to the very large exhibit high $\tau$ whereas those where size is homogeneous exhibit low $\tau$. Accordingly, $r_y$ and $r_x$ are the distance between the realisation of $y$ or $x$ and the corresponding median $\mathcal{Y}$ or $\mathcal{X}$. As in the previous example, observed $r_y, r_x$ may differ but they should be viewed as realisations of a common, underlying $r$ so that scale invariance is verified.

The ratio of the medians, $\mathcal{Y}/\mathcal{X}$, is the median ratio, being also the natural scaling factor between $y$ and $x$. If, for example, the median Sales in a given industry is £1,000 and for Earnings it is £100 whereas, for firm A, both Sales and Earnings are $r$ standard deviations above or below this norm, then

$$\text{Earnings}_A = 100\ \exp[r\tau]$$

and

$$\text{Sales}_A = 1,000\ \exp[r\tau].$$

Now consider firm B which is more profitable, the same volume of Sales generating Earnings $n$ standard deviations above A, i.e.,

$$\text{Earnings}_B = 100\ \exp[(r+n)\tau] = \text{Earnings}_A\ \exp[n\tau]$$

whereas

$$\text{Sales}_B = \text{Sales}_A.$$

Again, not only is the ratio independent of the magnitudes of its components, the difference in profitability between B and A, which is shown above to be $\exp[n\tau]$, is also independent of $r$. Moreover, this difference in profitability also 'accrues' with the dispersion of size. It is possible, therefore, to compare changes in profitability across industries exhibiting distinct size patterns. By contrast, where ratio components evolve linearly with size, the traditional pre-requisite for the validity of ratios is satisfied but the measurement is influenced by the size of the firm and by the value of the norm.

It is thus demonstrated that the existence of a scaling factor is *not sufficient* to grant valid ratios. Components must also be exponential and, in that case, firm size may be expressed as a constant rate of change. The multiplicative character of accounting variables, largely supported by evidence, seems to validate this second postulate. Indeed, the Gibrat's law is widely accepted as a simplified mechanism capable of broadly describe how positive accounting numbers are generated. The following section shows that the consideration of randomness may not necessarily affect these findings.

# 3 Ratios of Random Variables

So far the paper has derived formulations leading to valid ratios, namely where both components of the ratio are driven by the same, constant rate of change. Although deviations from norms were included in the discussion, they were treated as isolated cases, not as random deviations. When explicitly considering randomness, some new questions must be addressed. This section first shows that random variables can be scale invariant. Then the statistical characteristics of firm size are outlined. Finally, the problem of differences or negative components is addressed.

## 3.1 Random Scale Invariance

The characteristics of (5) leading to valid ratios can easily be replicated in a random context. Suppose, for instance, that the *observed* behaviour of accounting variable $x_k$ may be described as

$$x_k = \mathcal{X}_k \exp[r_j \tau + Z], \tag{6}$$

where $r_j$ is the expected continuously compounding rate of change denoting the effect of firm size and $Z$ is the random component. According to (6), accounting variables are *lognormal* and, as before, any item taken from the

$j^{th}$ set of accounts will exhibit the same logarithmic expectation $r_j$ so that scale invariance is verified.

Ratios of such variables could be validly used. The problem with (6) is that this form cannot be derived from (4) with generality. Indeed, the simplest random generalisation of (4) would lead to descriptions such as

$$x_k = \mathcal{X}_k \exp[(r_j - \frac{\sigma^2}{2})\tau + Z], \tag{7}$$

rather than to (6). Since the compounding effect is now influenced by the standard deviation $\sigma$ of $Z$, not just by size, it follows that variables governed by (7) would not be proportional in spite of obeying (1). It is concluded therefore that (1) cannot express scale invariance with generality, namely in the case of random variables.

Formulations such as (1) fail to portray random proportionality because they pre-suppose that changes $dY$ or $dX$ are smooth[12] whereas smoothness cannot be postulated for random changes. There are, however, components of random mechanisms that evolve smoothly and scale invariance may arise from a smooth effect that co-exists with random effects. Indeed, the question here is not whether random scale invariance is possible but how it should be expressed structurally. Random proportionality must be possible since, e.g., for lognormal variables, a natural scaling factor, the median of the ratio, does exist. Actually, the median is often used as a norm in ratio analysis (Lev & Sunder, 1979).

The correct formulation of random scale invariance may be found by identifying mechanisms that lead to (6) rather than to (7). However, it may be easier to simply answer the question of how (1) should be transformed so as to become insensitive to the variance and covariance of $Y$ and $X$. If, for instance, (1) is transformed so as to portray the equality of two continuously compounding rates of change, a mechanism portraying random proportionality is obtained. Recall that $r_j$ relates to the corresponding continuously compounding rate by the transformation $\log(r_j + 1)$. When this is applied to both sides of (1) it becomes

$$\log \frac{Y + dY}{Y} = \log \frac{X + dX}{X}$$

which may be abridged to

$$d \log Y = d \log X. \tag{8}$$

It is easy to verify that the above is indeed a quite general formulation of scale invariance.[13] It applies to deterministic as well as random variables, either continuous or discrete. In the deterministic case and in most practical applications, the difference between (8) and (1) is of lesser importance. Where structural descriptions are required, however, only (8) is guaranteed to portray the equality of two rates of change capable of generating proportional variables.

Differences between (8) and (1) are now illustrated within the framework proposed by Tippett (1990), who used Stochastic Calculus,[14] to study the ratios of some continuous-time Markov processes, concluding that ratios will necessarily drift over time. Here, by contrast, it is found that ratios may not drift.

Suppose that components $y$ and $x$ of a ratio are governed by stochastic differential equations consistent with (8), i.e.,

$$d\log y = s_j d\tau + \sigma_y dz_y \qquad \text{and} \qquad d\log x = s_j d\tau + \sigma_x dz_x \tag{9}$$

where $d\log y$ and $d\log x$ stem from a deterministic effect, $s_j d\tau$ plus a stochastic effect, $\sigma_y dz_y$ or $\sigma_x dz_x$, specific to $y$ or $x$.[15] The deterministic effect is supposed to model the pervasive influence of firm size, being the same for items from the $j^{th}$ set of accounts so that scale invariance is verified, and is further assumed to be constant throughout the generative period so that changes are also size-independent. The summation of all $d\tau$, $\tau$, reflects the length of the accounting period during which the generation of $j$ takes place.

Using the Itô lemma[16] to exponentiate (9) the trivial result is

$$\frac{dy}{y} = (s_j + \frac{\sigma_y^2}{2})d\tau + \sigma_y dz_y \qquad \text{and} \qquad \frac{dx}{x} = (s_j + \frac{\sigma_x^2}{2})d\tau + \sigma_x dz_x$$

which, after integration, yields

$$y = \mathcal{Y}\exp(s_j\tau + \mathcal{Z}_y) \qquad \text{and} \qquad x = \mathcal{X}\exp(s_j\tau + \mathcal{Z}_x) \tag{10}$$

where levels $\mathcal{Y}$, $\mathcal{X}$ are initial values and $\mathcal{Z}_y, \mathcal{Z}_x$ are Wiener processes with variances $\sigma_y\tau$, $\sigma_x\tau$ respectively. $y$ and $x$, therefore, stem from the exponentiation of Brownian motions, being accordingly known as exponential or geometric Brownian motions. Notice how (10) is simply the stochastic version of (3), (5) or (6), preserving those characteristics whereby the postulates of ratio analysis are verified. Ratios of such variables evolve as

$$\frac{y}{x} = \frac{\mathcal{Y}}{\mathcal{X}}\exp\mathcal{Z}, \tag{11}$$

| | Exponential Brownian Motion | Doléans exponential (Tippett, 1990) |
|---|---|---|
| $d\log x$ | $sd\tau + dz$ | $(r - \sigma^2/2)d\tau + dz$ |
| $dx/x$ | $(s + \sigma^2/2)d\tau + dz$ | $rd\tau + dz$ |
| $x$ | $\mathcal{X}\exp(s\tau + Z)$ | $\mathcal{X}\exp[(r - \sigma^2/2)\tau + Z]$ |
| $E(x)$ | $\mathcal{X}\exp[(s + \sigma^2/2)\tau]$ | $\mathcal{X}\exp(r\tau)$ |
| ratio | $\mathcal{R}\exp Z$ | $\mathcal{R}\exp[(\sigma_y^2 - \sigma_x^2)\tau/2 + Z]$ |
| $E$(ratio) | $\mathcal{R}\exp[(\sigma_r^2/2)\tau]$ | $\mathcal{R}\exp[(\sigma_y^2 - \rho\sigma_y\sigma_x)\tau]$ |

Table 1: Simplified formulations tested for valid ratios in this paper and in Tippett (1990). $Z$ has an expanding variance.

not only removing $s_j\,\tau$, the effect of size, but also exhibiting no inherent drift. $\mathcal{Z}$ is also a Wiener process with variance $(\sigma_y^2 + \sigma_x^2 - 2\,\rho\,\sigma_y\,\sigma_x)\,\tau$, $\rho$ being a (residual) correlation coefficient between $z_y$ and $z_x$.

Table 1 shows the differences between the generative mechanism used in this example and that in Tippett (1990).[17] Thus the apparent drift in ratios is just a consequence of the choice of the model.

It may be helpful to note that ratios of Markov processes, albeit valid, are useless except in the short-term. This is because the ratio method requires a comparison (in this context between values in different periods) and any such comparison using Markov processes carries no information at all except in the one-period comparison, in which case drifts would be negligible. In the analysis of Markov series, therefore, the existence of drifts would not make matters any worse than they are already. Moreover, by definition, non-stationary variables should drift over time irrespective of being ratios (in exponential Brownian motions the drift stems from the combined effect of an expanding variance and the exponentiation). In short, Tippett's (1990) choice of context and model, and the subsequent criticism of the ratio method, illustrates the potential danger of an *a priori* exclusion from the reasoning of the possibility that ratios may be valid.

In summary, the validity of the ratio method is indeed plausible in the case of random variables. When modelling the effect of size on the generation of accounting numbers, continuous compounding should be preferred to formalisms aimed at describing sampling processes, as appropriate. The widely used exponential Brownian motion adheres to such requirement. The following section discusses the modelling of firm size, showing how to interpret it as a statistical effect.

## 3.2 Multiplicative Formulations and Size

Sales, Earnings, Assets and other accounting variables are routinely viewed, in domains such as Industrial Economics, as multiplicative, i.e., broadly lognormal. In spite of this, until recently the accounting literature has discarded lognormality as incompatible with sound reasoning while quoting the influential fallacy introduced by Eisenbeis (1977),[18] or the fact that some ratios are Normal, or even the distinction between accounting stocks and flows. Excessive skewness and other characteristics of multiplicative variables were interpreted as distortions of normality or as a side effect of non-proportionality (Barnes, 1982).

The case for multiplicative mechanisms was made by McLeay (1986*b*), where it was explained that, in variables which are accumulations of similar transactions, lognormality is the natural assumption. Tippet (1990) followed, using multiplicative forms in the time domain. This section shows that lognormality is indeed important, not only because it provides the appropriate viewpoint to study ratios. More than that, lognormality entails the assumption that firm size is a pervasive statistical effect present in accounting numbers.

First, recall why accounting variables cannot be described as resulting from the kind of additive process that underlies Normal variables. Whilst each transaction contributing to the amount reported as, say, Total Sales for a given period is itself a random event, an individual transaction contributes to the reported aggregate not in a manner which could lead to either an increase or decrease in Total Sales, but by accumulation only. Accumulations of random events tend to be multiplicative, as opposed to additive, because the likelihood of realisations is conditional on the occurrence of a chain of several previous events. Such likelihood thus stems from multiplying, rather than adding probabilities. Other economic phenomena such as wealth or stock prices are also multiplicative but nowhere are reasons for this behaviour so neatly evident as in the case of Sales, Assets and other accounting numbers. The difference between accounting stocks and flows or the existence of accounting numbers that may take on negative values should not be viewed as capable of undermining the above reasoning, as explained in the following section.

Descriptions of the inter-relationships amongst effects greatly differ between additive and multiplicative variables. For additive data, distributions are preserved when variables are added or subtracted. This is not the case for multiplicative data where distributions are preserved when variables are multiplied or divided. Accordingly, the simplest additive formulation is $x = \mu + Z$, where $x$ is explained as an effect $\mu$ (the expectation), plus a random deviation, $Z$ while the multiplicative equivalent is $\mathbf{x} = \mathcal{X}w$, where a realisation of $\mathbf{x}$ is explained as the product of a level, $\mathcal{X}$, and a random factor $w$.

The likelihood that $x$ may stretch beyond two or three standard deviations above or below $\mu$ is very small. Therefore, in general, additive formulations may describe deviations from an average size but they are inadequate to describe large differences in size. By contrast, in the case of multiplicative formulations, the exponential nature of $w$ leads to likely values of $\mathbf{x}$ over a much wider range. This is why lognormality (or other forms of multiplicative behaviour) often denotes a size influence whereas normality generally denotes a size-free variable. In turn, the presence of size means a population where growth is in progress, whereas size-free variables denote a stable population where growth is no longer observed. It is thus somehow contradictory to accept, as some authors actually do, that accounting numbers are lognormal but, at the same time, failing to model the corresponding influence of size on those numbers.

The examination of conditions for the valid use of ratios has the advantage of leading naturally to formulations where firm size is present. In the case of the two ratio components $\mathbf{y}$ and $\mathbf{x}$ from the $j^{th}$ set of accounts, (6) may be written in logarithmic form as

$$\log \mathbf{y} = \mu_y + s_j\tau + Z_y \qquad \text{and} \qquad \log \mathbf{x} = \mu_x + s_j\tau + Z_x \tag{12}$$

where $\mu_x = \log \mathcal{X}$ and $\mu_y = \log \mathcal{Y}$. The above are known as *components of variance* formulations of the *mixed model* type. They account for differences in relation to the expectation where such differences are introduced by discrete sources of variability. The effect of firm size, $s_j\tau$ is, for both models, the *random* effect and $\mu_y$, $\mu_x$ are the *fixed* effects.

It was mentioned that the simplest additive formulation, $x = \mu + Z$, describes deviations in relation to $\mu$. When $x$ is explained, not only by the expected value but also by $d$, a component of the variance of $x$, then the ensuing model, $x = \mu + d_j + Z$ is similar to those in (12) where $d_j$ is the expected deviation from $\mu$ introduced by the $j^{th}$ level of $d$. If, as in (12), the same effect is present in two variables, it is possible to remove it from measurement by subtracting variables. Size, therefore, plays the role of a component of the variance of the logarithms of accounting variables. The different sets of accounts that may be considered in a given sample are the levels of such effect.

Components of variance are distinct from other, better known models, namely those incorporating co-variances. In the face of existing research where it was insisted that co-variance terms should be included in the modelling of the logarithms of accounting variables, it is important to state clearly that there is no theoretical underpinning supporting the generalised use of co-variances in this context and that such use would be, in most of the cases, inadequate.[19]

Ratios formed with components obeying (6) may be used validly under broad conditions. In a cross section comparison the median of the ratio (which can be estimated by the geometric mean) is the scaling factor for $\mathbf{y}/\mathbf{x}$ and the measurement is also size-independent. In a time series analysis it makes no difference whether $\mathcal{Y}/\mathcal{X}$ is the ratio in a previous period, a benchmark, or a prediction: the measurement will obey or not the second postulate depending solely on the nature of the processes $Z_y$, $Z_x$. In practice, an expanding variance and other types of non-stationarity often found in accounting series (Wu, Kao & Lee, 1996), may dissuade analysts from comparing values separated by more than one year. As stressed, this limitation is caused by non-stationarity, not by ratios.

The superior degree of definition brought about by (12) may allow a more focused discussion of the statistical characteristics of reported numbers while providing a basis upon which improvements may be built. For instance, (12) clearly suggests how to estimate the size of a firm based on a given set of accounts. Once size has been estimated, it is possible to create ratios capable of removing just size, or to seek a precise description of the co-variance structure of accounting variables. The following section explains how accounting differences and negative values should be viewed.

### 3.3 The Distribution of Ratios

One major source of scepticism regarding the usefulness of ratios has been the diversity of their statistical distributions: some ratios are broadly Normal, others are skewed positively or negatively.[20] McLeay (1986*b*) proposed a framework for understanding the distribution of ratios where components are supposed to be either multiplicative (accumulations) or additive (differences). The diverse distributions found in ratios would stem from the different combinations of these two cases. Tippett (1990) adhered to this view. Trigueiros (1995) contended that all accounting variables should be viewed as multiplicative, i.e., positively skewed, showing how apparently normal or negatively skewed distributions stem from the bounding effect of

| Ratio | Example | Transformation to use | Boundaries of the ratio |
|---|---|---|---|
| R $= Y/X$ | Current Ratio | $\log R$ | $0, \infty$ |
| R $= (X-Y)/X$ | Sales Margin | $\log(1-R)$ | $-\infty, 1$ |
| R $= (Y-X)/X$ | Change in Capital Employed | $\log(1+R)$ | $-1, \infty$ |
| R $= (Y+X)/X$ | Interest Cover | $\log(R-1)$ | $1, \infty$ |
| R $= X/(Y+X)$ | Liabilities Ratio | $\log(1/R-1)$ | $0, 1$ |
| R $= X/(Y-X)$ | Leverage Ratio | $\log(1/R+1)$ | $0, \infty$ |

Table 2: Some transformations of financial ratio based on accounting identities where $Y$ and $X$ are lognormal, positive only variables.

accounting identities (which preclude, for example, the ratio Total Debt to Total Assets from spreading out beyond the value of 1). The same author has provided transformations (table 2) able to bring these bounded ratios back to their multiplicative behaviour.

At present, papers generally incorporate the intuition that distributions of ratios may be influenced by accounting identities, but the suggestion that all accounting variables should be viewed as multiplicative has not been taken up. Instead, authors still adhere to the framework proposed by McLeay (1986*b*). For instance, Cooke & Tippett (2000) incorporate into their valuation models the bounding effect of accounting identities as proposed by Trigueiros (1995). McLeay (1997) and McLeay & Omar (2000) present sets of transformations based on bounding effects without abandoning the distinction between the two types of variables, additive and multiplicative. This section discusses the distribution of flows and other accounting differences inclusive of those that may take on negative values.

Authors seem to believe that a ratio of the form $dx/x$, such as the ratio expressing relative changes in Capital and others of the same type, should approach normality. This belief is based on the known fact that lognormal $x$ may lead to Normal $dx$. However, this is so only where $dx$ are small in comparison to $x$. Now, for accounting data this is simply not the case. Accounting differences or flows such as changes in Capital lead to multiplicative ratios because, in most instances, these flows are not small by any possible measure, not least when compared with the respective stock.

Accounting differences should not be viewed as a typical '$dx$'. Actually, they are not much different from stocks, both being large accumulations.

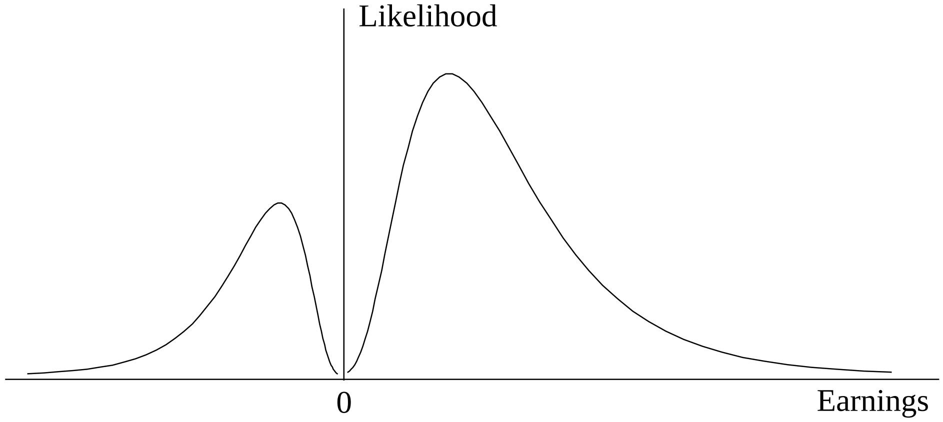


Figure 1:

Annual Sales, for instance, in spite of being a flow, often is larger than stocks such as Fixed Assets, e.g., in high turnover industries.

Similarly, numbers that may take on negative values should not be viewed as possessing additive, nearly normal distributions, but simply as subtractions of synchronised accumulations, i.e., as the result of subtracting two lognormal, positive–only, variables, both generated under the effect of the same firm's size. The density function of one such subtraction is easy to simulate using (12). It turns out that such distribution is not unique. Rather, it is a juxtaposition of two approximately lognormal density functions, one for positive and the other for negative values, as depicted in figure 1.

The assumption of a Normal or similar distribution would imply that the reporting of immaterial profits or losses should be *more likely* than that of material losses. In general, however, losses, as well as profits, are proportionate to the size of the firm thus the reporting of immaterial losses is indeed less likely than that of size-related losses. That is, the probability density of profits must decrease and then increase again after passing through zero. Actually, this is what the simulated distributions in figure 1 predict. By contrast, Normal and related distributions are unable to account for the specific behaviour of small numbers.

The consequences for ratio analysis are that, in general, ratios are multiplicative no matter their components' type, stocks or flows. When the numerator of the ratio is linked by some accounting identity to its denominator, the appropriate transformation (Table 2) will bring the ratio back to a standard behaviour. As for ratios where the numerator may take on negative values, it seems as though there is no other choice but to examine two different populations, one for positive and the other for negative realisations. This, after all, is how practitioners generally deal with negative ratios (Lev & Sunder, 1998).

To conclude, most accounting numbers should be viewed as accumulations. Flows such as Earnings and other variables where negative cases are possible simply reflect the subtraction of two accumulations driven by the same influence of size. Their statistical behaviour is broadly multiplicative, negative values being distributed in a way that resembles the *mirror–image* of a lognormal density function.

## 4 Conclusions

Only after understanding the reasons for the success of financial ratios can their limitations be put into a proper perspective. Accordingly, this study has examined two postulates underlying valid ratio analysis, extracting from them a set of conclusions, summarised as follows:

- the components of ratios are scale invariant (proportional) and are generated under homogeneous relative growth;
- the effect of firm size is an expected continuously compounding rate of change specific to each set of accounts;
- flows or other differences stem from subtracting two multiplicative variables, both influenced by the same size effect;

These conclusions constitute a foundation as they provide a simplified model of the behaviour of accounting variables upon which improvements may be built.

In setting out the above insights into the theoretical foundations of the ratio method, the paper has addressed the question as to whether the use of financial ratios is motivated by tradition rather than by well-founded considerations. It is our conclusion that, in spite of their simplicity, ratios are governed by an explicit set of conditions and, moreover, they require data of the type that has been shown to characterise the figures reported in company accounts.

Given this, it may then be asked why so many previous contributions to the literature on financial ratios have led to such a pessimistic view of ratio analysis. Reasons seem to relate to an apparent lack of theoretical drive in the part of most authors, probably fed by the conviction that accounting data is too complex, being impossible to find regularities there. These authors, typically, neither accept the working hypothesis that ratios may be valid nor produce other hypotheses, thus remaining stuck in a sceptical trap or at the mercy of assumptions whose implications they do not seem fully aware of. In both cases conclusions are hampered by such scarcity or inadequacy of assumptions.

It is by no means a coincidence that scepticism about the usefulness of ratios usually goes hand in hand with other manifestations of scepticism, namely about the possibility of defining firm size exactly. By contrast, efforts to understand why ratios are useful have led to a clearer view of other related subjects, specifically the role that firm size plays on the generation of those numbers.

The lack of theoretical drive leads, in turn, to the difficulty in distinguishing what is truly damaging for the financial measurement from what is negligible. Typical examples are the concern about 'intercept terms' or that with drifts in the time domain. The often-quoted statement that almost all of the assumptions required for valid ratio analysis are likely to be violated in practice (Lev & Sunder, 1979), should be viewed in the same light. The statement is surely correct but it might as well be applied to Newton's Laws of Motion and to many other models considered as good approximations in normal circumstances. Distortion, in spite of its presence in mathematical models, may have a negligible effect on measurement.

Moreover, assumptions may be violated without invalidating a methodology. When weighing inaccuracy against the ability to provide an intuitive interpretation with a parsimonious model, it may well be that such a trade-off could prove to be largely favourable to the less accurate methodology. This kind of trade-off is particularly relevant to the ratio method. Ratios, having just one degree of freedom, are able to measure deviations from constant proportions. The condition of proportionate growth, rather than a limitation, is a direct consequence of this: one unique parameter is only able to deal with scale invariant changes, i.e., the modelling of the common changes of both components. Now, by providing deviations from norms and nothing else, ratios offer in a succinct form the information that financial analysts seek. Conversely, analysts would find it difficult to use models where the relevant information is scattered in several parameter values.

Thus, to conclude, the challenge facing research into financial ratio analysis is not how to increase the complexity of models. Rather, it is how to take account of the limitations of the parsimonious ratio model without changing the specific characteristics of the measurement.

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## 6 Notes

1. 'Multiplicative', 'proportionate', 'exponential' and 'lognormal' are terms variously used in the literature on the growth of firms to designate a family of skewed distributions related to proportionate growth (the Gibrat Law).

2. The statement stresses that, notwithstanding the potential to mislead

in well known cases such as in the presence of seasonality or where both the numerator and the denominator may take on negative values, ratios have no *general* source of invalidity associated with their use.

3. When ratios are used to make predictions (Whittington 1980), a comparison also takes place, the object of interest being an observed discrepancy as well. For instance, when predicting Profits using the Profit Margin ratio, the reference is the margin taken as the norm and the discrepancy is the proportion of predicted Profits which is not explained by Sales.

4. That is, ratio analysis entails a *measurement*.

5. Only measurements are valid or invalid, the problem of the validity of ratios being thus circumscribed to cases involving a comparison. Indeed, any financial indicator whose full interpretation would not require a comparison, thus being a fundamental magnitude in its own right, would be neither valid nor invalid.

6. Scaling factors are multipliers by definition and will not easily yield to the role of expected values which, also by definition, are additions. The expected ratio of two Normal variables, for instance, depends on the magnitudes of components. The expected ratio of two lognormal variables depends on the variance and co-variance of components. The case, documented in Lev & Sunder (1979) and other authors, of an error term whose variance is proportional to the square of the numerator of the ratio, would lead to homeoskedastic, constant ratios but this transformation bears no resemblance to the way accounting numbers are generated.

7. This term, borrowed from texts on linear regression, is often found in texts on the vlidity of ratios.

8. Lev & Sunder (1978, p. 190) implicitly refer to this formulation when they notice that ratios assess both the marginal and the average effect of a change in $Y$ on $X$.

9. Williamson (1984), for instance, identified 11 ratios used by *Fortune 500* companies as part of their annual reports. Sales appears in 4 of these ratios, Total Assets in 3, Net Income and Operating Income in 2. Amongst this set of 11 ratios, only 2 do not share components with other ratios.

10. Using the expansion

$$df(y,x) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy,$$

changes $d\,(y/x)$ experienced by the ratio $y/x$ may be expressed as

$$d\,\frac{y}{x} = \frac{y}{x}\left(\frac{dy}{y} - \frac{dx}{x}\right)$$

and the existence of actual discrepancies in relation to the scaling factor implies, in the general case, that

$$\frac{dy}{y} \neq \frac{dx}{x}.$$

This, together with the postulate that valid ratios are size-independent, leads to the conclusion that $d\,(y/x)$ is size-independent with generality where both $dy/y$ and $dx/x$ are size-independent. The case where $dy/y - dx/x$ is size-independent but individual $dy/y$ and $dx/x$ are size-related, although worth examining, lacks generality.

11. The role of $\tau$ in cross section and in time series is thus made similar. In both cases $\tau$ allows for comparisons between cases where the time elapsed or the dispersion are not the same.

12. Smoothness of $dY$ or $dX$ is what allows extrapolation amongst contiguous values of $Y$ or $X$. Extrapolation, in turn, is required to make $dY$ and $dX$ as small as desired. Such requirement, called *infinitesimal convergence* is the basic postulate of Calculus.

13. The increase in generality brought about by (8) in relation to (1) mirrors that obtained by the use of the geometric mean as a summary measure in order to overcome adverse distribution conditions.

14. Stochastic Calculus is an analytical technique applicable to continuous time Markov processes (*diffusions*). In spite of being non-differentiable, these processes nevertheless allow, in most of the cases with practical interest, the manipulation of the stochastic equivalent to differential equations.

15. Random terms $dz_y, dz_x$ are limits of increments of Wiener processes $\mathcal{Z}_y, \mathcal{Z}_x$ as the time interval approaches $d\tau$. For practical purposes

$dz_y = Z_y\sqrt{d\tau}$ and $dz_x = Z_x\sqrt{d\tau}$ where $Z_y$ and $Z_x$ are time-independent standard Normal random variables. Wiener processes are zero mean, time-independent Normally distributed continuous Markov processes.

16. The Itô lemma is the tool of Stochastic Calculus. Given a stochastic function $F(x, t)$, the total derivative may be calculated using

$$dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dx + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(dx)^2.$$

Readers may find a useful introductions in the context of Financial Economics in Campbell, Lo and MacKinley (1997, pp. 339–349).

17. Another example where the distinction between processes in table 1 is relevant is the pricing of options when the variance of the underlying security is estimated using sampled returns whereas pricing models are developed on the basis of continuously compounded returns. Where sampled returns $dP/P$ equate $\mu d\tau + \sigma dz$, this is equivalent to $d\log P = (\mu - 1/2\sigma^2)d\tau + \sigma dz$, where the corresponding continuously compounded returns $d\log P$ are explained in terms of parameters $\mu, \sigma$ used in the previous process, which were estimated from the data (see, e.g., Campbell, Lo & MacKinley, 1997, p. 962).

18. Eisenbeis (1977) mistakenly stated that 'log-transformed variables give less weight to equal percentage changes in a variable where the values are large than when they are smaller [...thus] the implication [of using a logarithmic transformation] would be that one does not believe that there is as much difference between a \$1 billion and a \$2 billion size firms as there is between a \$1 million and a \$2 million size firms. The percentage difference in the log will be greater in the latter than in the former case' (p. 877). Eisenbeis' pitfall is that the calculation of proportions of the log-transformed measurement is equivalent to calculating proportions twice. This inappropriate warning against logarithms gave support at the time to the use of *ad hoc* techniques such as those proposed by Frecka & Hopwood (1983), and may have dissuaded researchers from attempting to fit adequate models to the distribution of ratios.

19. When logarithmic co-variances are used (Tippett, 1990; Cooke & Tippett 2000) it is pre-supposed that the relationship between two accounting variables cannot be linear except in singular cases. Although

some relationship between accounting variables may indeed be non-linear, the assumption that all relationships *must* be non-linear would be unnecessarily restrictive.

20. It may be mentioned, amongst many other examples, that So (1987) did not find positive skewness in the ratios Total Debt to Total Assets ($TD/TA$), Net Worth to Total Assets ($NW/TA$) and Current Assets to Total Assets ($CA/TA$), the latter even being negatively skewed. Watson (1990) also noticed that ratios $TD/TA$ and $NW/TA$ were symetrical. See e.g., Trigueiros (1995) for a review.