The use of Accounting Data in Predictive Models

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Any practitioner will tell you that accounting data is difficult to model. When, for instance, building a regression, you will find that data is plagued with influential cases capable of distorting estimation. Moreover, heteroscedasticity, fat tails and asymmetric distributions make your P-values meaningless. When trying to use the usual recipes to sort out some of these problems, you will find that most of the remedies simply do not work. Indeed, it is difficult to make any sense of accounting data.

Yet, accounting data is not unpredictable. It obeys clearly defined rules. Once these rules are understood, it becomes possible to use accounting variables in parametric models, either in the form of ratios or directly as raw numbers.

The goal of this note is to facilitate the understanding of the characteristics of accounting data so that it can be used in statistical models. Specific guidelines are offered on how to transform ratios into well-behaved variables, how to deal with non-proportionality, how to accurately model firm size, how to model Earnings and how to solve other difficulties.

Accounting Numbers are Multiplicative

Accounting data is the numerical information found in annual reports of firms. Reports contain sets of accounts (the Profit and Loss Account, the Balance Sheet and others) and item in these sets will report on a specific magnitude. The volume of sales of the year, for instance, is reported in the item Sales. Magnitudes found in sets of accounts are the raw material for financial analysis. But before being of any use, these numbers must be combined to form ratios. Typically, two numbers from the same report, say Earnings and Net Worth, may be chosen as the numerator and denominator of a ratio. Most accounting data used in statistical models is in the form of ratios.

It is impossible to understand the statistical characteristics of ratios without understanding first the characteristics of the numbers they are made of and the way they interact. Therefore, the first section of this note is devoted to such numbers.

1 The first and most important fact about numbers found in accounts is that they cannot be described as resulting from the type of random mechanism that leads to Normal variables. Normal variables stem from additive random mechanisms: an observed distribution is the result of adding a large number of other distributions. For instance, the distribution of weight in adults stems from adding the probabilities associated with genetic effects, eating habits and other effects. In the limit, any addition of probability distributions, no matter which, leads to the Normal distribution (the Central Limit theorem).

Obviously, in an additive mechanism, each intervening effect may lead to an increase or to a decrease in the likelihood associated with the resulting event. Adequate eating habits may, for instance, be able to balance a genetic pre-disposition to put on weight, lowering the expected weight.

2 The mechanism generating accounting numbers is different: effects always reinforce each other. This is because such effects are, in this case, every individual transaction contributing to a reported magnitude. Indeed, each transaction contributing to the amount reported as, say, Total Sales for a given period, is itself a random event. It contributes to the reported number, not in a manner which could lead to either an increase or decrease in Total Sales, but by accumulation only.

Accumulations of random events lead to multiplicative, as opposed to additive variables, because the likelihood of realisations is conditional on the occurrence of a chain of previous events, not on any free interaction of influences, some positive, others negative. Such likelihood thus stems from multiplying rather than adding probabilities.

3 Multiplicative distributions are easy to recognise. They are skewed, exhibiting long tails towards positive values. As a consequence, some of the observations in a sample are likely to exhibit very large magnitudes in comparison with others, thus giving the impression of being outliers.

Contrasting with additive distributions where skewness and kurtosis are independent, in mechanisms of the multiplicative type both statistics are manifestations of a unique, underlying phenomenon, variability. Therefore, highly volatile variables exhibit markedly skewed and leptokurtic distributions whereas those where variability is small have almost symmetrical, non-kurtoptic distributions.

4 Accumulation is just one amongst several processes leading to multiplicative numbers. Any variable where magnitude x is, in average, proportional to changes dx, will exhibit a multiplicative distribution. The natural form[1] in the origin of such type of variable is

$$\frac{dx}{x} = \hat{x} d\tau + dz \tag{1}$$

where $\hat{x}$ is an expected percent change, τ is the variable supposed to drive changes in x, dz is a small random disturbance and $\hat{x} d\tau$ is supposed to be independent of τ. Therefore, in this type of variable, percent changes are additive. What approaches normality as a limit is percent change, not absolute change.

The mechanism depicted in (1) is known as the “Gibrat’s Law of Proportionate Effect”. It leads to lognormal distributions, that is, distributions where the logarithm of observations is Normal, or to other types of multiplicative distribution. Multiplicative, proportionate, exponential and lognormal are terms variously used to designate the family of skewed distributions with its origin in (1). Aitchison and Brown (1957) describe the lognormal distribution.

5 Sales, Earnings, Assets and other accounting aggregates are since long known, in domains such as Industrial Economics and others, to be multiplicative, i.e., broadly lognormal. In spite of this, until recently the accounting literature has discarded lognormality in accounting data as incompatible with sound reasoning while quoting the influential fallacy introduced by Eisenbeis (1977),[2] or the fact that some ratios are apparently Normal, or even the existence of negative values in some accounts. Excessive skewness and other characteristics of multiplicative variables were interpreted as distortions of normality or, in the case of ratios, as a side effect of non-proportionality (Barnes, 1982). In the Accounting research domain, the case for multiplicative mechanisms was made by McLeay (1986) and Trigueiros (1995).

6 The peculiar characteristics of lognormal variables must be borne in mind in any context involving the manipulation of these variables or their ratios. Lognormality cannot be treated as a simple departure from normality. For coefficients of variation[3] beyond 0.25, skewness and kurtosis are so severe that most observations concentrate in a small region with only a few extreme values spreading out over a wide range. No parametric tool is robust enough to avoid severe distortion when such data is used.

7 Statistical models try to find functional forms that are capable of reflecting relationships amongst effects. Descriptions of the inter-relationships amongst effects greatly differ between additive and multiplicative variables. For additive data, distributions are preserved when variables are added or subtracted. This is not the case for multiplicative data where distributions are preserved when variables are multiplied or divided. The addition or subtraction of two Normal variables will be Normal; the product or ratio of two lognormal variables will be lognormal. The simplest additive formulation would be

$$x = \hat{x} + z \tag{2}$$

where x is explained as effect $\hat{x}$, the expectation, plus a random deviation z. The multiplicative equivalent would be

$$\log \mathbf{x} = \hat{x} + z \text{ or } \mathbf{x} = \mathcal{X} w \tag{3}$$

where $\mathbf{x}$ is now explained as the product of a constant magnitude, $\mathcal{X}$, by a random factor w.

8 The likelihood that x may stretch beyond two or three standard deviations above or below $\hat{x}$ is very small. Therefore, in general, additive variables describe deviations from an average magnitude but they are unable to describe large differences in magnitude. By contrast, in the case of multiplicative formulations, the exponential nature of w leads to likely values of $\mathbf{x}$ over a much wider range. The volume of sales of United Biscuits, a firm in the 95^{th} size percentile of its industry, is about 500 times that of firms in the 5^{th} size percentile. Additive observations would never be able to model such huge discrepancies. This is why lognormality often denotes size influence whereas normality generally denotes a size-free variable.

How to Use Financial Ratios in Statistical Models

The existence of a common size influence in magnitudes reported in the accounts of firms led to the use of ratios. Ratios such as Return on Equity, Interest Cover, Debt to Net Worth and many others, are widely used by managers, practitioners and analysts. They control for this common effect of size so that comparisons may be made.

9 Where accounting numbers are lognormal, then ratios should be lognormal as well. But some ratios show unexpected characteristics, which make them difficult to use in statistical models. For instance, although most ratios are indeed lognormal, Total Debt to Total Assets, Net Worth to Total Assets and others are apparently Normal. Current Assets to Total Assets is negatively skewed (see, e.g., So, 1987). How is this possible?

The reason is straightforward. Accounting identities preclude some ratios from taking on all the values a skewed distribution would allow. This constraining effect is clearly observable when plotting, on a logarithmic scale, the two components of a ratio. Figure 1 shows, on the left, the effect of a constraint imposed by Total Assets on the spread of Net Worth and, on the right, a relationship where accounting identities play no role.

10 Adequate transformations can take into account constraining mechanisms yielding unconstrained ratios. For example, any ratio where the numerator cannot be larger than the denominator, i.e.,

$$\frac{x_i}{\sum x_i} \tag{4}$$

can be transformed into the corresponding, unconstrained, ratio

$$\frac{x_i}{(\sum x_i) - x_i}. \tag{5}$$

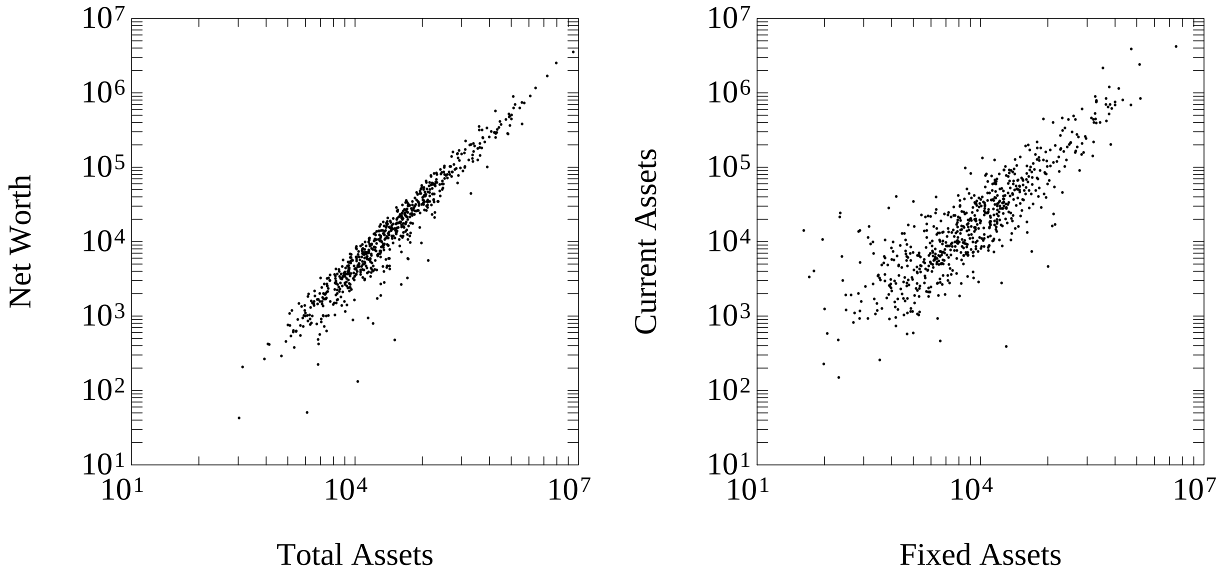

Figure 1: Scatter-plots of constrained (left) and unconstrained (right) bivariate distributions.

Case No	Ratio	Example	Transformation to use	Boundaries of the ratio
1	$\mathrm{R} = Y/X$	Current Ratio	$\log R$	$0, \infty$
2	$\mathrm{R} = (X - Y)/X$	Sales Margin	$\log(1 - R)$	$-\infty, 1$
3	$\mathrm{R} = (Y - X)/X$	Change in Capital Employed	$\log(1 + R)$	$-1, \infty$
4	$\mathrm{R} = (Y + X)/X$	Interest Cover	$\log(R - 1)$	$1, \infty$
5	$\mathrm{R} = X/(Y + X)$	Liabilities Ratio	$\log(1/R - 1)$	$0, 1$
6	$\mathrm{R} = X/(Y - X)$	Leverage Ratio	$\log(1/R + 1)$	$0, \infty$

Table 1: Transformations for constrained ratios (McLeay and Trigueiros, 2003).

The unconstrained ratio corresponding to the ratio Fixed Assets to Total Assets (FA/TA) is the ratio FA/CA where $CA = TA - FA$. The information contained in both ratios is the same. The difference between them is just functional.

11 Table 1 summarises transformations able to bring ratios affected by several types of constraints into parametric behaviour. Ratios where there is no constraint are lognormal.

12 Awareness of constraining mechanisms and the way they affect ratios removes one major obstacle in understanding and using financial ratios. But not all is explained. A fact that remains unaccounted for is the existence of fat tails (leptokurtosis) in the logarithms of all types of ratios, that is, even after appropriate transformations are applied. Such leptokurtosis, though, is not severe and may, in most cases, be ignored.

The following section discusses the distribution of accounting flows (differences such as Earnings) which may take on negative values thus posing particular difficulties to modeling.

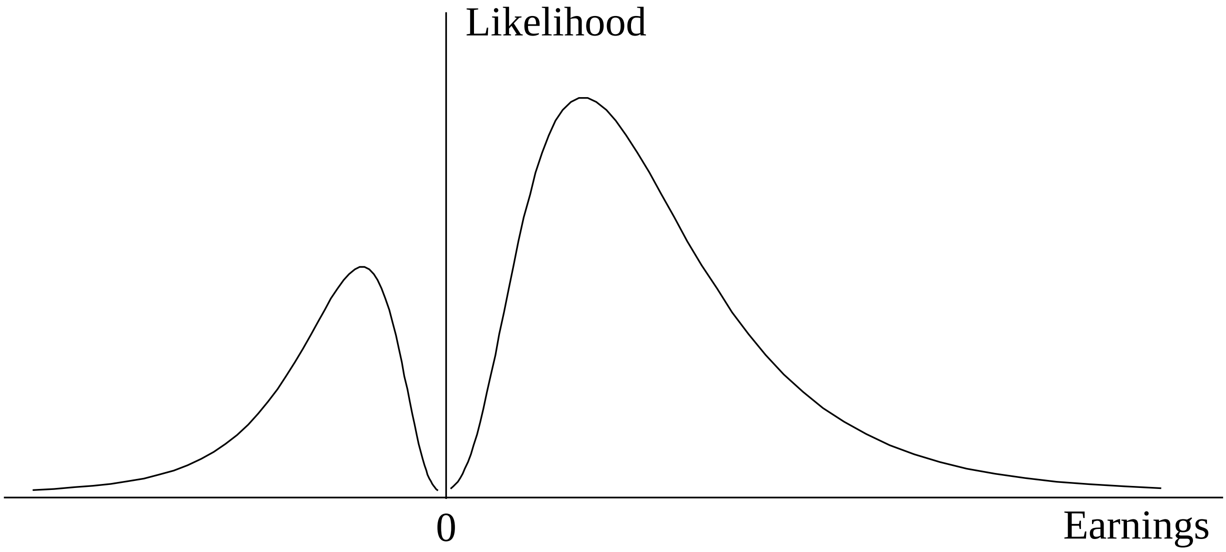

Figure 2: The density function of Earnings is a juxtaposition of two lognormals.

The Distribution of Earnings

The literature on the distribution of ratios seems to consider profits as additive, albeit not necessarily Normal (McLeay, 1986; Tippett, 1990; amongst others). Probably this is because authors focus on profitability ratios, not on profits figures.

13 Profitability ratios basically express percent changes in Net Worth. Authors believe that a ratio of the form dx/x should approach normality since lognormal x lead to Normal dx where dx are small disturbances. However, in order to qualify as a small disturbance, dx must be really small when compared with x. Now, for most numbers taken from sets of accounts, this is simply not the case. Flows such as changes in Net Worth are not small enough to qualify as a disturbance, not least when compared with the respective stock. Annual Sales, in spite of being a flow, often is larger than stocks such as Fixed Assets, e.g., in high turnover industries. Accounting flows should not be viewed as typical dx. Actually, they are not that different from stocks, both being large accumulations. Thus profitability and other ratios are indeed multiplicative.

14 Numbers that may take on negative values are simply the result of subtracting positive-only accumulations. A magnitude reported as Earnings is obtained by subtracting the different types of costs and expenses from Revenues. The distribution of these variables should therefore stem from subtracting lognormal distributions. The task of analytically determining such distributions is not easy as it requires working out the logarithm of a subtraction. There is however a fact that simplifies analysis. Costs and Revenues are correlated because both are influenced by the same effect, size. When correlation is taken into account it becomes possible to approximate analytically the distribution of flows. It turns out that, for conditions typically found in industries, such distribution is not unique. Rather, it is a juxtaposition of two approximately lognormal density functions, one for positive and the other for negative values, the latter being a mirror-image of the lognormal as depicted in figure 2. Simulation confirms this result.

15 Ratios formed with such distributions may be markedly two-tailed, giving the impression that they are near symmetry. Fat-tailed distributions such as Student's t or Cauchy's may indeed fit them closely (McLeay, 1986). It should be made clear, however, that the hypothesis of additive distributions leads to unreasonable conclusions. The reporting of immaterial values must be less likely than that of size-related values. The probability density of Earnings, for instance, must decrease when approaching zero and then increase again after passing through zero into negative values as predicted by Figure 2. This is because losses, as well as profits, must be proportionate to size. Additive distributions would imply that the reporting of immaterial profits or losses is more likely than that of material losses.

Ratios are multiplicative no matter their components' type, stocks or flows. When the ratio is constrained by some accounting identity, an appropriate transformation will bring it back to a standard behaviour. Specifically, profitability ratios will benefit from transformation no. 3 in Table 1. Indeed, most commonly found situations involving negative flows are solved simply by using this or other transformation. As for ratios where the denominator, not just the numerator, may take on negative values, it seems as though there is no other choice but to consider two populations, one for positive and the other for negative denominators. This, after all, is how practitioners deal with such ratios.

How to Account for Non-Proportionality in Ratios

Measurement using ratios requires proportionality between components. If the natural relationship between ratio components y and x is of the form $y = a + bx$ (non-proportional) rather than $y = bx$ (proportional), then the measurement will be misleading as y/x cannot have constant standards or norms (see, e.g., Lev and Sunder, 1979). Evidence on non-proportionality in some specific ratios is provided by Sudarsanam and Taffler (1995) and others.

16 How to overcome problems posed by non-proportionality? The Law of Proportionate Effect acknowledges that changes dx in (1) may be proportional, not to x itself, but to $x - \delta$. In this case, the natural form governing the generation of reported magnitudes will be

$$\frac{dx}{x-\delta} = \hat{x}d\tau + dz \tag{6}$$

instead of (1). Where δ is constant, conditions leading to lognormal x in (1) generate, in (6), distributions known as three-parametric lognormal with threshold δ (Aitchison and Brown, 1957). Three-parametric lognormal density functions are simply the result of displacing lognormal distributions by δ (figure 3). In a time series, for instance, the existence of fixed costs may lead to a constant displacement in the distribution of Operating Costs.

17 In order to cope with non-proportionality, one of the following ratios should be used

$$\frac{y-\delta_y}{x} \quad \text{or} \quad \frac{y}{x-\delta_x} \tag{7}$$

for, respectively, three-parametric lognormal numerators or denominators. Thresholds are δ_y or δ_x. Indeed, there exists a value of such δ for which the above, non-proportional, ratios become proportional.

18 How damaging thresholds are for the ratio measurement? Due to the exponential character of (6), reported magnitudes may attain values many times larger than the threshold and, in such case, $x - \delta_x \approx x$. Non-proportionality is significant only where magnitudes are not much larger than δ, for example, in the time-series context. Indeed, comparatively large size-independent thresholds are plausible only in such context. Observations, in a time-series, have their origin in the same object, one firm in different periods. Values which are constant inside firms, such as fixed costs, may create comparatively large thresholds. In cross-section, as observations have their origin in different objects, size-independent thresholds would require the existence of industry-wide 'fixed costs'. Since such costs should allow for the survival of small firms,

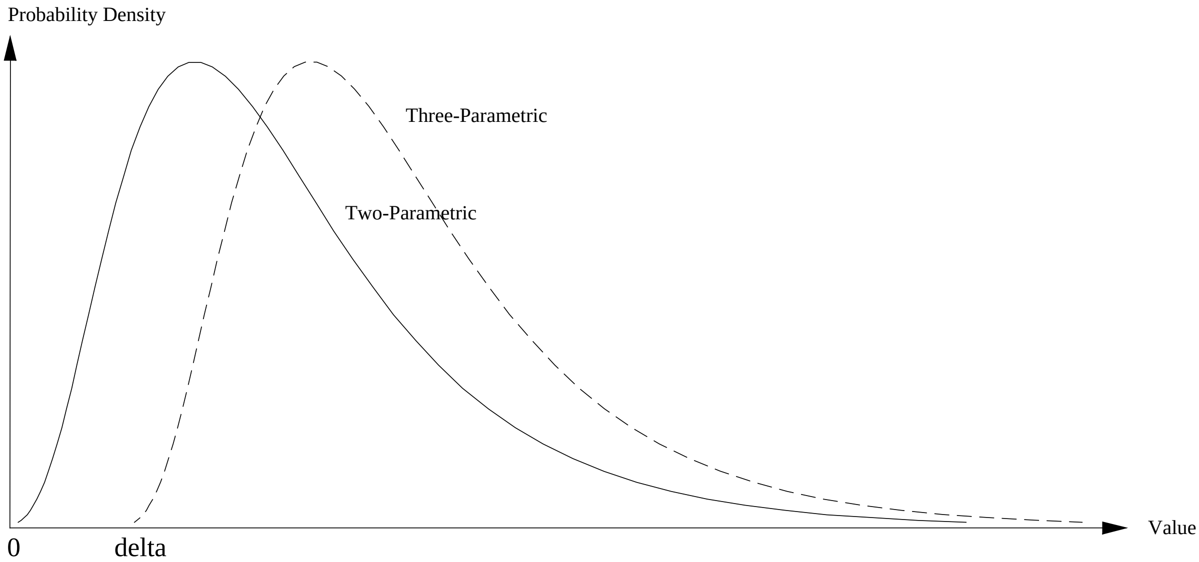

Figure 3: The lognormal (solid line) and the three-parametric distributions (dashed line).

they must be small. An industry-wide cost of £4m for food manufacturers in the UK would be only 0.2% of United Biscuits' revenues, but it would equal or exceed the turnover of the 5% smaller firms in the industry.

19 Fixed costs are also likely to generate size-related thresholds, namely in cross section where large firms have large fixed costs and small firms have small fixed costs. In this case δ is similar to any other accounting variable. According to (6), x will be larger than expected for comparatively large δ, (e.g., the case of large firms) and $\hat{x}$ is not constant. As a consequence, size-related thresholds do distort ratio measurement.

Notice that the problem, in this case, is not any displacement in the distribution of ratio components. Since δ are small for small x and large for large x, distributions are not displaced. The problem is non-linearity in their relationship.

20 Size-related thresholds require the use of ratios of the type

$$\frac{y}{x^{\beta}}. \tag{8}$$

For a specific value of β, the ratio will have a constant standard thus allowing measurement. On a logarithmic scale, the functional form of such measurement is

$$\log y - \beta \log x = \hat{r} + z \tag{9}$$

where $\hat{r}$ is the logarithm of the ratio standard and z is the observed deviation from that standard. (9) is similar to a regression. The slope, β, is approximate to the unit in the case of strict proportionality. Slopes smaller than 1 denote a negative δ. In cross-section, they bias large firm's ratios downwards, mimicking scale effects.

21 How should the two types of δ just outlined (constant and size-related) be estimated? The ratio Fixed Assets (FA) to Current Assets (CA) is now used in a cross-section example. Five models of ratio, as follows, are compared. Each model is presented together with its logarithmic counterpart as multiplicative formulations require logarithmic scaling prior to coefficient estimation. Figure 4 shows, on a logarithmic scale, how the usual ratio (solid line) compares with each model. Figure 5 is a replica of Figure 4 on the original scale (only the region near the origin is displayed).

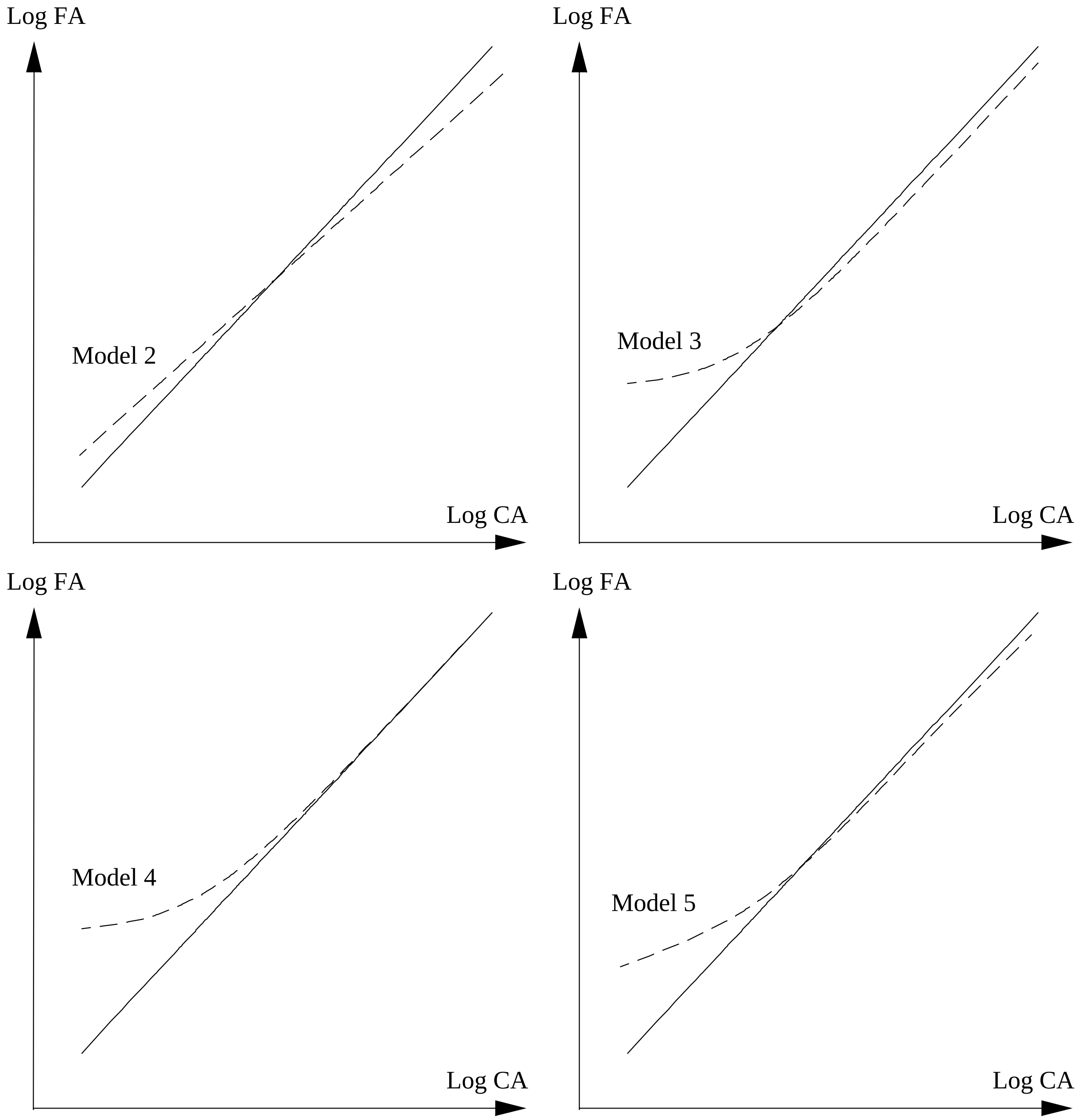

Figure 4: The usual ratio (solid line) compared, on a logarithmic scale, with the slope ratio (Model 2), threshold ratios (Models 3 and 4), and the threshold plus slope ratio (Model 5).

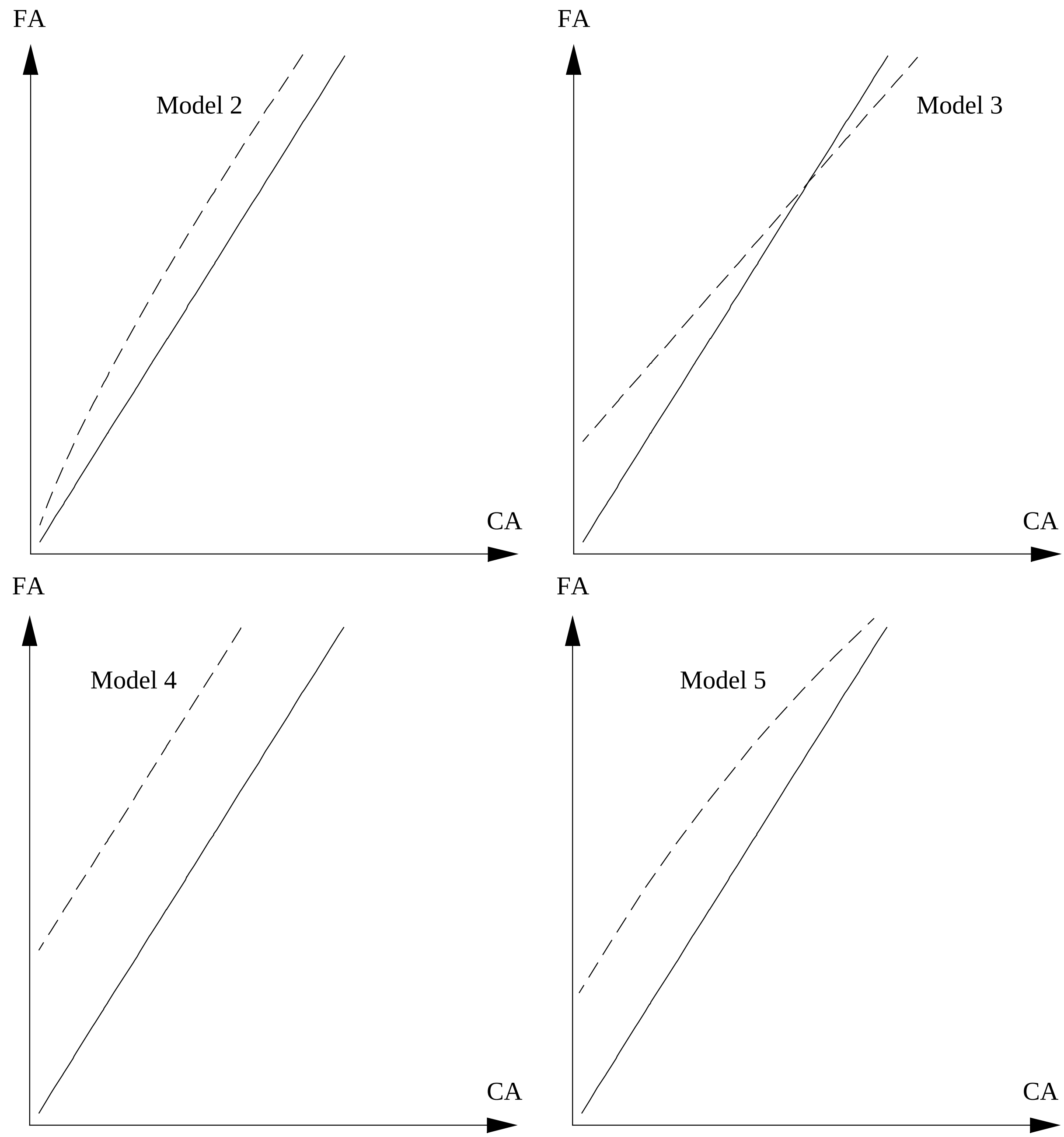

Figure 5: The usual ratio (solid line) compared with the slope ratio (Model 2), threshold ratios (Models 3 and 4), and the threshold plus slope ratio (Model 5).

Model 1: The usual ratio, no correction introduced. This ratio requires the estimation of one parameter, the standard. In cross-section, an appropriate standard is the median of the distribution of the ratio (on a logarithmic scale, μ). Therefore,

$$\frac{\mathrm{FA}}{\mathrm{CA}} = 10^{\mu} w \quad \text{or} \quad \log \mathrm{FA} = \mu + \log \mathrm{CA} + z \tag{10}$$

with z being a random disturbance and $w = 10^{z}$. When the distribution of CA and FA are nearly lognormal, the standard may be estimated by subtracting the averages of $\log FA$ and $\log CA$. However, since a significant threshold in the distribution of CA is ignored, such standard will be misleading.

Model 2: A ratio with a size-related threshold in the denominator. This ratio may control for economies of scale and other non-linearities. It requires the estimation of two parameters, the standard (μ in logarithms) and the slope β as in (8). A regression similar to (9) may be used to estimate both:

$$\log \mathrm{FA} = \mu + \beta \log \mathrm{CA} + z \quad \text{corresponding to the ratio} \quad \frac{\mathrm{FA}}{\mathrm{CA}^{\beta}} = 10^{\mu} w \tag{11}$$

Graphically, this ratio is a straight line on a logarithmic scale and, as mentioned, it is non-linear on an ordinary scale. A simple least-squares regression may estimate the standard and β.

Model 3: Constant threshold in the denominator, jointly estimated parameters. This ratio corrects for a constant threshold but in such a way that the standard is corrected as well. It requires the estimation of two parameters: the standard (μ in logarithms) and δ, the constant threshold. Both are estimated using

$$\log \mathrm{FA} = \mu + \log(\mathrm{CA} - \delta) + z \quad \text{corresponding to the ratio} \quad \frac{\mathrm{FA}}{\mathrm{CA} - \delta} = 10^{\mu} w. \tag{12}$$

Graphically, this ratio is non-linear on a logarithmic scale and a straight line on an ordinary scale.

Model 4: The threshold ratio with independently estimated parameters. In this case there is only one degree of freedom, μ, the logarithm of the ratio standard. The constant threshold of the distribution of CA, d, is estimated prior to that of the standard. μ is thus estimated using

$$\log \mathrm{FA} = \mu + \log(\mathrm{CA} - d) + e \quad \text{corresponding to the ratio} \quad \frac{\mathrm{FA}}{\mathrm{CA} - d} = 10^{\mu} w. \tag{13}$$

Graphically, the model is non-linear and parallel with model 3 on a logarithmic scale and a straight line on an ordinary scale. This ratio probably is the most useful as non-proportionality is accounted for albeit converging with the usual ratio for medium-sized and large firms.

Model 5: Ratio with both constant and size-dependent thresholds. This ratio requires the estimation of three parameters, μ, β and δ, using

$$\log \mathrm{FA} = \mu + \beta \log(\mathrm{CA} - \delta) + z \quad \text{corresponding to} \quad \frac{\mathrm{FA}}{(\mathrm{CA} - \delta)^{\beta}} = 10^{\mu} w. \tag{14}$$

Graphically, the ratio is a mixture of models 2 and 3.

The independent estimation of a constant threshold (model 4) may be carried out using any of the available procedures to detect three-parametric lognormality in distributions. In this example, $\delta_{CA} = -£320,000$ is estimated using the procedure suggested by Royston (1982).[4] The joint estimation of δ and the ratio standard (models 3 and 5) may be carried out using iterative least-squares algorithms.

Model	μ	β	δ	R^2	Skewness	Kurtosis
1	-0.46			72%	0.27	2.18
2	-0.21	0.94		76%	0.07	1.30
3	-0.49		-£528,000	78%	-0.12	1.19
4	-0.46		-£320,000	76%	-0.12	1.78
5	-0.40	0.99	-£403,000	78%	-0.12	1.19

Table 2: Parameters and statistics obtained with the five models.

TESCO PLC	Results	
1983–1987	Regression	Threshold Ratio
Model used for estimation	Costs $= a + b$ Sales $+ z$	log Sales $= \mu + \log(\text{Costs} - \delta) + z$
Estimated coefficients:	$a = £149m,\ b = 0.91\ (R^2 = 99\%)$	$\mu = 0.036,\ \delta = £125m\ (R^2 = 99\%)$
Functional form obtained:	$\frac{\text{Costs} - £149\text{m}}{\text{Sales}} = 0.91$	$\frac{\text{Costs} - £125\text{m}}{\text{Sales}} = 0.92$

Table 3: Estimating Fixed Costs with a regression and a threshold ratio.

Table 2 shows the variability explained (R^2) by each of the five ratios. Skewness and kurtosis of residuals on a logarithmic scale (z) is also displayed. As can be seen, by allowing β into model 2, R^2 approaches the variability explained by constant thresholds (models 3 and 4). Once δ is accounted for, β returns to an estimated value of nearly 1 (model 5).

22 The second example is taken from Steele (1989). It shows how to correct for non-proportionality in time series. During the period 1983–1987, Operating Costs and Sales of TESCO, a retailer, are

Year	1983	1984	1985	1986	1987
Operating Costs	£2,211m	£2,518m	£2,911m	£3,216m	£3,407m
Sales	£2,276m	£2,594m	£3,000m	£3,355m	£3,593m
Ratio	97.15%	97.07%	97.03%	95.85%	94.84%

Regressions where Operating Costs explains Sales are traditionally used to estimate Fixed Costs (the intercept term). Table 3 shows the functional forms used and the results obtained by a regression and by the threshold ratio. Estimates are clearly different in spite of the fact that the relationship is almost deterministic ($R^2 = 99\%$). The slope of the regression, $b = 0.91$, is thus near the standard of the ratio, $10^{\mu} = 0.92$. Should the correlation between Sales and Operating Costs be smaller, then b would be clearly smaller than 10^{μ}. In the limit, for a correlation approaching zero, b would also become zero and a, the supposed estimate of Fixed Costs, would equal the expected value of Operating Costs. Regressions are thus inadequate for the task. They introduce in the estimation the spurious effect of correlation. By contrast, the threshold ratio correctly explains Fixed Costs as a displacement in the distribution of Operating Costs.

How to Model Firm Size

Firm size is often chosen as a substitute for numerous theoretical constructs, ranging from risk to liquidity or even political costs. Size is also an ingredient of its own in many theoretical models. In spite of this widespread use, size has remained a poorly defined concept. Where the use of size is required by theory, empirical studies typically revert to using proxies such as Total Assets, Market Capitalisation or Sales (see, e.g., Bujaki and Richardson, 1997).

23 The multiplicative character of magnitudes reported in accounts suggests, as stressed before, that the generation of their distribution is driven by size. It is indeed possible to deduce a simple and effective definition of firm size from the two following assumptions: first, reported magnitudes broadly obey the Gibrat's Law of Proportionate Effect; second, financial ratios do remove the effect of size.

24 In order for ratios to be effective, the likelihood of observed discrepancies in relation to the standard must be independent of size. For instance, the Return on Assets ratio is useless if an increase or decrease of 2% has different meanings for small and large firms. The probability distribution of ratios must therefore be homeoscedastic in terms of size. This implies, in the general case, that percent changes in both the numerator and denominator must be size-independent. As mentioned, variables where percent changes are size-independent are said to obey the Gibrat's Law of Proportionate Effect. Indeed, the widespread use of ratios agrees with the fact that reported numbers are multiplicative.

It was also mentioned that multiplicative mechanisms lead to broadly lognormal distributions. In fact, observed x generated as in (1) may be described functionally as

$$x = \mathcal{X}[1+\hat{x}]^{\tau+z} \text{ for discrete } \tau \text{ or } x = \mathcal{X}\exp[\hat{x}\tau+z] \text{ for continuous } \tau. \tag{15}$$

where τ is the variable which drives changes in x, $\hat{x}$ is a logarithmic expectation and z is a random increment. The level $\mathcal{X}$ is the value of x for $\tau = 0$. It may be demonstrated that only where continuous compounding is assumed, as in the right-hand side of (15), may ratios be validly used.[5]

25 As for the second assumption mentioned above, that of ratio y/x eliminating the effect of size, it leads to the requirement that the numerator y and denominator x should both be generated under the effect of equal rates of change ($\hat{y} = \hat{x}$). In fact, where $\hat{y} \neq \hat{x}$, the ratio standard would not be constant, showing a rate of change of $\hat{y}/\hat{x}$ with τ. This requirement importantly suggests that, not just y and x but the other numbers found in a specific annual report, are generated under the effect of the same rate of change. Since the validity of the ratio method rests on the validity of several, widely used ratios, not just on one or two cases, and given that numbers used to form one of such ratios are also used to form other ratios, then there must exist a common source of variability underlying numbers reported in the accounts of a firm in a given year. Where numbers $x_1, x_2, \cdots, x_k, \cdots$ all belong to a specific annual report, this assumption leads to $\hat{x_1} = \hat{x_2} = \cdots = \hat{x_k} = \cdots$. It is possible to show that such unique source of variability possesses the attributes of size.[6]

26 How can size be estimated from this unique rate of change? A specific annual report, say, report j, is characterised by what value τ assumes. x_k, the magnitude reported by item k, is explained as

$$x_k = \mathcal{X}_k \exp[\hat{x}_j\tau + z], \tag{16}$$

where $\hat{x}_j\tau$ is the effect of size (the same for all items reported in j). In an additive form,

$$\log x_k = \mu_k + \sigma_j + z \tag{17}$$

where $\mu_k = \log \mathcal{X}_k$ and $\sigma_j = \hat{x}_j\tau$. Formulation (17) is basically an Analysis of Variance, i.e., a type of linear model aimed at explaining variability in terms of membership of discrete classes. Specifically, in (17) $\log x_k$ is explained by its membership of two classes, the item class, μ_k, and the annual report class, σ_j. The item class is a fixed (deterministic) effect, as it denotes the fact that k is a specific item amongst those in the sets of accounts reported by firms. These accounts are indeed fixed in number and in type. By contrast, the annual report's class is a random effect: it denotes the fact that j is one of the (randomly selected) annual reports in the sample. Each of these two classes possesses levels, namely, there can be as many levels of k as items in the sets of accounts; there can be as many levels of j as different reports in the sample.

27 In cross section, μ_k is the expected value of logarithmic magnitudes reported in item k. It is estimated as the mean of k calculated using all the annual reports in the sample. Thus, in this case, $\mathcal{X}_k$ in (16) is the median of magnitudes reported in k. The size effect, σ_j, is the expected $\log x_k - \mu_k$ for numbers in j and its estimation is straightforward: given N numbers, all of them reported in j, (17) is first applied to each of these numbers. The N formulations obtained are then added. Since σ_j is the same in all of these formulations, it is possible to write

$$\sigma_j = \frac{1}{N}\sum_{k=1}^{N}(\log x_k - \mu_k) - \frac{1}{N}\sum_{k=1}^{N} z_k .$$

Any source of variability common to all $\log x_k$ in j is, by construction, accounted for by σ_j. Therefore, even where correlation amongst some z may exist, the term

$$\frac{1}{N}\sum_{k=1}^{N} z_k$$

should tend to zero with an increasing N, leading to

$$\text{estimate of } \sigma_j = \frac{1}{N}\sum_{k=1}^{N}(\log x_k - \mu_k)\,. \tag{18}$$

An estimated size may thus be obtained simply by averaging the logarithms of appropriately adjusted magnitudes drawn from the firm's actual report. Exact confidence intervals for σ_j can also be obtained, the corresponding standard errors being t-distributed with $N-1$ degrees of freedom.

28 The estimation of σ_j faces two obvious difficulties. First, x_k in (18) cannot be randomly drawn from all possible accounts because items such as Earnings, being a subtraction, may take on negative values and cannot be transformed into logarithms. This pre-selection of items introduces a bias in the estimation of size. Second, inside each annual report, some z are correlated. This increases the standard error.

In practice, size can be estimated with good accuracy as an average of the logarithms of positive-only magnitudes found in items such as Cash and Short Term Investments, Receivables, Total Inventories, Property, Plant and Equipment (net), Common Equity, Number of Employees, Net Sales or Revenues, Cost of Goods Sold (excluding Depreciation), Depreciation, Depletion and Amortisation of the year, Interest Expense on Debt and others.

29 It may be argued that magnitudes reported in the accounts of firms may possess two or three size dimensions, not just one. Notwithstanding the cases of recently formed firms or other less general cases, the hypothesis of size being multi-dimensional seems difficult to sustain.[7]

Tools for the Graphical Examination of Accounting Data

The superior degree of definition brought about by (17) allows a focused examination of the statistical characteristics of reported numbers while providing a basis upon which improvements in financial measurement may be carried out. Indeed, (17) has suggested how to estimate size. Once an accurate size measurement is available, it is possible to create ratios capable of removing just size, or to seek a precise description of the co-variance structure of accounting data.

30 This section describes one such improvement, the Rotated Residual Plot (RRP), aimed at increasing the usefulness of ratios. Notice first that z in (17) is the logarithm of a ratio in which the denominator is size and the numerator is the deviation of the magnitude reported in item k from expectation. Such ratio thus reflects the proportion by which reported magnitudes diverge from expected in firms of that size. The z are, in fact, size-free Cash, Receivables, Inventories and so on.

Since pairs of numbers convey two-dimensional information and ratios are just one variable, when ratios are used instead of their components some information is lost. Not only information about size. Ratios also discard size-free information potentially useful. Given (17), the size-free information conveyed by the ratio y/x can be written in logarithms as the subtraction

$$z_y - z_x = \log \frac{y}{x} - \mu_{y/x}, \tag{19}$$

Two axes, Y and X, are now defined so that z_y is measured along the Y-axis and z_x is measured along the X-axis. All the size-free information conveyed by the ratio components will be represented by a point, $\{z_y, z_x\}$. A 45^o, anti-clockwise rotation is now applied to Y and X. Such rotation is performed by transformation

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{20}$$

After rotation, the new X-axis measures $z_y - z_x$ and the new Y-axis measures $z_y + z_x$. As mentioned, (19), $z_y - z_x$ is, in logarithms, the information evidenced by y/x. Since the new Y-axis is orthogonal to the ratio, all the size-free information not evidenced by the ratio will be evidenced by the new Y-axis, that is, by $z_y + z_x$. On an ordinary scale, $z_y + z_x$ is the ratio xy/s^2. Its denominator, s^2, is the squared effect of size ($\sigma = \log s$). Ratios y/x and xy/s^2 thus evidence the two orthogonal dimensions of the same size-free information. Ratio xy/s^2 is thus a complement for y/x.

31 Is this new type of ratio potentially interesting? Ratios remove size in two ways (Lev and Sunder, 1979):

- Explicitly, when the denominator is selected simply because it is a good proxy for size. Ratios explicitly removing size are meant to measure whether a particular magnitude is large or small when compared with the size of the firm.
- Implicitly, when the denominator of the ratio is selected so as to produce a desired contrast with the numerator. Ratios remove size implicitly when they are meant to compare a magnitude with another magnitude, irrespective of size.

Though, in practice, the separation between these functions is not always distinct, some ratios are clearly meant to remove size explicitly, while the majority is meant to evidence a contrast, removing size only implicitly. For example, in the two ratios Working Capital to Total Assets and Current Assets to Current Liabilities, the former assesses liquidity by comparing Working Capital with a proxy for size, while the latter compares short-term assets with short-term liabilities, regardless of size. The complement xy/s^2 removes size explicitly. When used alone, its usefulness will be circumscribed to specific instances. But when used together with the ratio it may become useful as discussed next.

32 The Rotated Residual Plot (RRP) is a scatter-plot in which the X-axis shows (on a logarithmic scale) deviations of y/x from the industry median. This axis thus evidences how a characteristic of the firm, such as liquidity or profitability, compares with the industry norm. The Y-axis shows, also in logarithms, deviations of xy/s^2 from the industry median. Recall that the RRP is a rotation of scatter-plot of z_y with z_x. As seen, this leads to a new X-axis showing $z_y - z_x$ and a new Y-axis showing $z_y + z_x$.

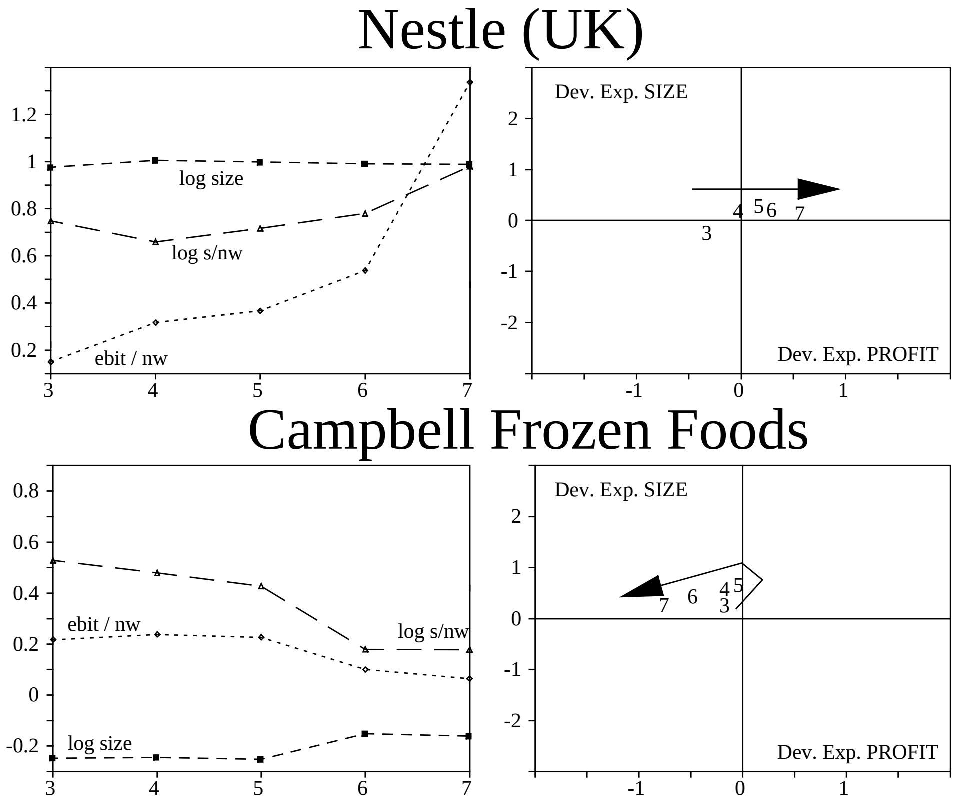

Figure 6: On the left, $EBIT/NW$ and size during five years (marks 3 to 7). On the right, the RRP.

33 The following example uses the RRP to examine profitability of Food Manufacturers in the UK. The ratio to be used is Earnings Before Interest and Tax ($EBIT$) to Net Worth (NW), supposed to evidence efficiency of the firm's worth (percent profit generated by unit worth). The sample contains the reports of food manufacturers during five years. An estimated size, $\sigma = \log s$, is first obtained by averaging, inside each annual report, the logarithms of six magnitudes previously mean-adjusted year by year. Specifically,

$$\begin{aligned} \sigma \quad = \quad 1/6 & \quad (\log \text{Sales} + \log \text{Wages} \\ & + \quad \log \text{Number of Employees} + \log \text{Debtors} \\ & + \quad \log \text{Current Liabilities} + \log \text{Current Assets}). \end{aligned} \tag{21}$$

Next, the Y and X coordinates in the RRP are calculated. If

$$z_{ebit} = \log EBIT - \overline{\log EBIT} - \sigma, \qquad \text{and} \qquad z_{nw} = \log NW - \overline{\log NW} - \sigma, \tag{22}$$

then the two axes of the RRP are:

$$Y = z_{ebit} + z_{nw} \qquad \text{and} \qquad X = z_{ebit} - z_{nw} \tag{23}$$

$\overline{\log EBIT}$ and $\overline{\log NW}$ are industry averages.

34 Figure 6 shows the RRP of two firms together with the corresponding time-histories of $EBIT$ to NW and $\sigma = \log s$. The X-axis of the RRP shows, in logarithms, percent deviations from the average profitability in the industry. The Y-axis shows, on the same scale, how Earnings and Net Worth diverge from expected for size. The nearer a firm is to the centre of the RRP, the less it diverges from the industry norms for profitability and size. The first quadrant of the RRP shows firms with both Earnings and Net Worth above the average for size. The second quadrant means Net Worth above the average for size but Earnings below average, and so on. Trajectories parallel to the Y-axis show firms growing in capital and in profits but not in efficiency. Trajectories parallel to the X-axis show firms gaining to the industry in efficiency while keeping capital in line with size.

The RRP is interesting because unfavourable, as well as favourable positions, are clearly evidenced as problems of size, not just efficiency. During the initial three years of the period, CAMPBELL increased its proportion of Net Worth to size. Since $EBIT$ also increased in the same proportion, the ratio was blind to this anomaly and only denounced the plunge in efficiency that ensued. During the same period, profitability at NESTLE increased relative to the industry. The RRP says as much but also points out that such gain was obtained purely by an increase in efficiency. The proportion of Net Worth and Earnings to size was kept near the industry average.

Concluding Remarks

Accounting data obeys a set of simple rules. The first rule states that, since reported numbers and ratios are multiplicative, logarithmic transformations should be used to bring data to a proper additive behaviour. The second rule explains that accounting identities often distort otherwise perfectly multiplicative distributions of ratios into unexpected shapes. Such identities should be accounted for prior to logarithmic transformation. The third rule explains the different ways to account for non-proportionality in ratios. The fourth rule states that numbers in a specific report are generated under the influence of size, being thus possible to obtain a size estimate just by averaging several of these numbers, conveniently adjusted. The distribution of Earnings and the development of two-dimensional tools for analysis were also discussed.

It is important to always bear in mind two important facts about ratios. First, besides those distribution-related difficulties which were addressed in this note, the use of ratios requires caution in order to avoid other, well-documented limitations: ratios produce ambiguous results when denominators, not only numerators, may take on negative values; ratios are sensitive to atypical magnitudes (numbers from the Profit and Loss account must be corrected for reported periods different from one year; accounts from the Balance Sheet may be distorted by seasonal effects).

The second important fact is that, notwithstanding the above limitations, ratios do not deserve the negative image conveyed by scientific journals and text-books. In spite of their widespread use, ratios are regarded with scepticism by scholars and are described as some primitive tradition with no scientific support. One major cause of such scepticism is the mentioned diversity in statistical distributions. Another is the fact that some reported numbers may take on negative values, which is difficult to reconcile with multiplicative behaviour. The note has addressed these two difficulties.

It may be asked why so many contributions to the literature have led to a pessimistic view of ratios and accounting data in general. Reasons seem to relate to an apparent lack of theoretical drive, probably fed by an *a priori* conviction that accounting data *should* be complex and full of exceptions, just as the production of such data is indeed complex and full of exceptions. It is expected that, by using suggestions from this note, researchers and practitioners may find that such *a priori* conviction is, after all, a tradition with no scientific support.

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Notes

1. Natural forms describe mechanisms rather than observations. They are often expressed as differential or as difference equations.

2. Eisenbeis (1977) mistakenly stated that 'log-transformed variables give less weight to equal percentage changes in a variable where the values are large than when they are smaller. The implication would be that one does not believe that there is as much difference between a $1 billion and a $2 billion size firms as there is between a $1 million and a $2 million size firms. The percentage difference in the log will be greater in the latter than in the former case' (p. 877). Eisenbeis' pitfall is that the calculation of proportions of the log-transformed measurement is equivalent to calculating proportions twice. This

inappropriate warning against logarithmic transformations gave support at the time to the use of *ad hoc* techniques such as those proposed by Frecka and Hopwood (1983).

3. The coefficient of variation is the standard deviation expressed as a fraction of expected value.

4. Royston uses trial and error to find out which δ maximises the parameter W in the Shapiro and Wilk's test of normality.

5. (1) may not necessarily lead to (15). Indeed, the simplest formulation generated by (1) is

$$x_k = \mathcal{X}_k \exp[(r_j - \frac{\sigma^2}{2})\tau + z], \tag{24}$$

rather than (15). Since the compounding effect is now influenced by σ, the standard deviation of z, not just by size, variables obeying (24) are non-proportional and cannot be used to form ratios. If, however, $\hat{x_1} = \hat{x_2} = \cdots$ is interpreted as an equality of continuously compounding rates of change, then a proportional mechanism is obtained.

6. Consider two annual reports, b and a. These reports are realisations of the same $x_1, x_2, \cdots$ and changes from a to b may be written

$$dx_1 = x_{1b} - x_{1a} \qquad dx_2 = x_{2b} - x_{2a} \qquad \cdots \tag{25}$$

Thus, from (1), it is possible to express magnitudes reported in b in terms of those in a as

$$x_{1b} = x_{1a}(1 + r_{ba}) \qquad x_{2b} = x_{2a}(1 + r_{ba}) \qquad \cdots \tag{26}$$

where r_{ba} is the percent change whereby any magnitude reported by a specific item in b differs from the magnitude reported by the corresponding item in a. Therefore, structurally at least, if one item is larger in b than in a, it follows that the other items in b will also be larger than the corresponding items in a and conversely. If, say, Current Assets is larger in b than the corresponding Current Assets in a, then Sales will also be larger in b than in a and so on. It is thus possible to say without ambiguity that set b is, as a whole, larger or smaller than set a. Now if it is possible to rank by size two annual reports, then it is possible to rank by size any number of reports. The common source of variability in $x_1, x_2, \cdots$, thus possesses the attribute of a size measurement.

7. While financial statements of large firms contain reported numbers that are many orders of magnitude larger than those reported in the accounts of small firms, inside each annual report magnitudes do not differ by as much. Yet, more than one size dimension would imply that, routinely, not in exceptional cases only, numbers with six or more digits should be found in the same report together with numbers with two or one digit. A ratio formed from items from non-agreeing dimensions would exhibit typical magnitudes of billions as well as micros. Such a ratio may indeed occur but only as the result of conditions not applicable generally. The fact that microscopic ratios or very large ratios are atypical, shows that items from the same annual report share, in general, a unique size influence. There are other reasons to support the conviction that each firm's actual size greatly influences the overall magnitude of numbers reported in its accounts. If variables such as Earnings were not closely related to size, then profitability and dividend yield would be diluted by any increase in size and firms would avoid growing.