SYSTEMATIC DRIFT IN QUOTIENTS OF CONTINUOUS-TIME STOCHASTIC PROCESSES AND IMPLICATIONS FOR FINANCIAL ANALYSIS

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ABSTRACT

The paper first shows that quotients of the type of diffusions widely used in Financial Economics exhibit systematic drift through time. This fact invalidates their use in Financial Analysis where scale-invariance is the basic requirement for valid ratio usage. The paper then develops a new definition of scale-invariance which is applicable to both stochastic and traditional calculus, showing that quotients of diffusions obeying such new definition no longer exhibit drift.

KEYWORDS

Financial Analysis, Stochastic Calculus.

INTRODUCTION

For reasons probably related to mathematical tractability, the literature on Financial Economics has paid great attention to continuous-time Markov processes, using the Itô lemma extensively as a tool to obtain useful closed-form solutions. The field of Financial Analysis is not an exception to this trend. In recent years, a considerable volume of research has emerged on the continuous time properties of financial ratios. The seminal work and procedures were laid down by Lev [2].

The present paper constructs a ratio of two Brownian motions and then studies the validity of such ratio in the light of the strict requirement upon which Financial Analysis is based.

The implicit assumption underlying the use of financial ratios Y/X is that the relationship between the numerator (Y) and the denominator (X) is expected to remain constant no matter the changes observed in the ratio components, i.e.

$$E(\frac{Y}{X}) = \text{Constant}.$$

For instance, the popular benchmark whereby the amount of Current Assets is expected to be twice as large as Current Liabilities is deemed to hold whether the figures involved are as large as \$2bn:\$1bn or as small as \$200:\$100. It is this which prompted authors such as Lev & Sunder [3] and Whittington [6] to state that the most basic requirement of financial ratio validity is expected proportionality between Y and X. Furthermore, these same authors also examined two conditions implicit in proportionality, namely a linear relationship between Y and X and the presence of a zero intercept term. Any form of systematic non-proportionality will render ratios useless for financial analysis since comparison between firms of different sizes would be misleading.

In order to conclude as to whether it is possible for ratios to be validly used, the first step consists of re-writing the above formula in a more general way. The above conditions for the validity of ratios emphasise the relationship between Y and X and not the way changes in Y should relate to changes in X so that the ratio remains constant. Yet it is obvious that, when Y is proportional to X, the rate of change of Y with respect to X remains constant and similar to the ratio itself. For ratios to be valid, therefore, the following relationship must hold:

$$\frac{Y}{X} = \frac{dY}{dX}$$

where dY, dX are any related changes or differences observed in Y and X. This formulation is the differential equivalent to Y/X = Constant. By re-arranging terms, the above identity becomes:

$$\frac{dY}{Y} = \frac{dX}{X} \tag{1}$$

Equality (1) shows that, implicit on proportionality, there is the condition of *scale invariance*, whereby the rate of change in Y should be equal to the rate of change in X. In other words, (1) says that ratios are valid only if the proportionate changes in the numerator and denominator are expected to be identical. For instance, when comparing firms in cross-section, if the Current Assets figure is expected to be three times larger in one firm than in another then this should also be the case for Current Liabilities or any other variable which is potentially useful as a component of a ratio. Similarly, in a time-series analysis, (1) implies that any variable eligible as a ratio component is expected to grow at the same rate. If, say, Sales is expected to grow by 12% in a given year, then Earnings should also be expected to grow by 12% during the year. It should not be surprising that the validity of ratios is conditional on the equality of proportionate changes in variables. Since ratios are scales, they are valid only where the scaling of the data makes sense and this implies scale invariance as a property of such data.

GEOMETRIC BROWNION RATIOS

Scale invariance, as defined by equality (1) applies to deterministic changes but, in general, it is not verified for stochastic changes in continuous-time Markov processes. Consider two financial aggregates y and x evolving as geometric Brownian motions,

$$\frac{dy}{y} = \mu_y dt + dz_y \quad \text{and} \quad \frac{dx}{x} = \mu_x dt + dz_x \tag{2}$$

where $\mu_y dt$, $\mu_x dt$ are instantaneous mean rates of growth of y and x respectively, dz_y and dz_x are white noise processes with variance σ_y^2 and σ_x^2. Applying Itô's lemma to the ratio

$$r = \frac{y}{x} \tag{3}$$

implies that the evolution of r is governed by the stochastic differential equation

$$\frac{dr}{r} = [\mu_y - \mu_x \rho\sigma_y\sigma_x + \sigma_x^2]dt + dz_y - dz_x \tag{4}$$

where $[\mu_y - \mu_x \rho\sigma_y\sigma_x + \sigma_x^2]dt$ is the instantaneous mean rate of growth in r whilst dz_y and dz_x are white noise. Note that the instantaneous variance of the rate of growth in r is:

$$\begin{aligned} \text{Var}[dz_y - dz_x] &= \text{Var}[dz_y] + \text{Var}[dz_x] - 2\text{Cov}[dz_y, dz_x] \\ &= [\sigma_y^2 + \sigma_x^2 - 2\rho\sigma_y\sigma_x]dt \end{aligned} \tag{5}$$

where Var[...] is the variance of the relevant random variable. Cov[...] is the covariance between the two random variables and ρ is the Pearson product moment correlation coefficient between dz_y and dz_x. The solution to the above stochastic differential equation is:

$$r = r_0 \exp[(\mu_y - \frac{1}{2}\sigma_y^2)t - (\mu_x - \frac{1}{2}\sigma_x^2)t + z_y - z_x] \tag{6}$$

where z_y, z_x are Wiener processes with zero mean and variance $\sigma_y^2 t$, $\sigma_x^2 t$. This implies that r is lognormally distributed with mean [1]:

$$E[r] = r_0 \exp[(\mu_y - \mu_x + \sigma_x^2 - \rho\sigma_y\sigma_x)t] \tag{7}$$

Since the expected value of the ratio is a function of time, scale invariance cannot be verified [4]. Even when $\mu_y = \mu_x$, some systematic drift in the ratio is expected.

MEAN-REVERTING RATIOS

In [2], it is argued that financial ratios may follow a partial adjustment model rather than a random walk. The distinguishing characteristic of this model is explained in [2] in the following terms:

> ... when the firm observes a deviation between its ratio and the industry mean.... it will adjust its ratio in the next period.... so that the observed deviation will be partially eliminated.

To illustrate the type of stochastic process which is consistent with this model, suppose that the two previously considered financial aggregates evolve in accordance with the following stochastic differential equations:

$$\frac{dy}{y} = -\lambda \log \frac{y}{g_t} dt + dz_y \tag{8}$$

and

$$\frac{dx}{x} = -\lambda \log \frac{x}{h_t} dt + dz_x \tag{9}$$

As previously, dz_y and dz_x are white noise processes with variance σ_y^2 and σ_x^2 respectively. These processes are elastic random walks in the sense that both y and x have a greater tendency to revert to their normal values of g_t and h_t, respectively, the further they are removed from them. In this respect $\lambda > 0$ is a parameter which measures the intensity with which the financial aggregates are drawn back to their normal values. Hence should $y > g_t$, then $\log[y/g_t]$ will be positive and the instantaneous expected rate of growth in y will be negative. Similarly, when $y < g_t$, the instantaneous expected rate of growth in y will be positive. Using (3) and Itô's lemma, it follows that the evolution of the ratio of these two financial aggregates is governed by the following stochastic differential equation:

$$\frac{dr}{r} = [(\sigma_x^2 - \rho\sigma_y\sigma_x) - \lambda \log \frac{r}{\mu_y}]dt + dz_y + dz_x \tag{10}$$

where $\mu_y = g_t/h_t$ is the 'normal' value to which the ratio reverts and ρ is the Pearson product moment correlation coefficient between dz_y and dz_x. Analysis similar to that employed in the previous section shows that the instantaneous mean rate of growth in the ratio amounts to:

$$[(\sigma_x^2 - \rho\sigma_y\sigma_x) - \lambda \log(r/\mu_y)]dt$$

Whilst the instantaneous variance is:

$$[\sigma_y^2 + \sigma_x^2 - 2\rho\sigma_y\sigma_x]dt$$

The solution to this stochastic differential equation is:

$$r = \mu_y \exp[\gamma_t - \frac{\sigma_y^2 - \sigma_x^2}{2\lambda}] \tag{11}$$

where:

$$\gamma_t = \mathrm{e}^{-\lambda t} \int_0^t \mathrm{e}^{\lambda s} dV \tag{12}$$

is an additive elastic random walk with:

$$dV = dz_y - dz_x.$$

(11) implies that r is lognormally distributed [1]. Further, it may also be shown [4] that the proportionality assumption implicit in financial analysis is also violated in this instance since co-variance introduced by the extra term present in the Itô formula means that same degree of non-proportionality will exist.

STOCHASTIC SCALE INVARIANCE

It should not be concluded from the above that it is impossible to observe scale invariance (and thence validly used ratios) in the expected relationship between stochastic variables.

The reason why (1) fails to encompass such type of variables is that it equates two real or effective rates of change (i.e. rates of change which are inherently discrete), whereas an equality between two continuous rates is now required. A formulation of the scale invariance condition which is robust regarding the nature of the variable (i.e. deterministic or stochastic, continuous or discrete) is obtained by equating expected continuous rates of change, as follows:

$$E\left[\log \frac{y + dy}{y}\right] = E\left[\log \frac{x + dx}{x}\right]$$

which may be abridged as

$$d \log y = d \log x. \tag{13}$$

In fact, suppose that the two ratio components y and x now are generated by the stochastic differential equations

$$d\log y_{ji} = s_j dt + \sigma_y dz_{y_i} \tag{14}$$

and

$$d\log x_{ji} = s_j dt + \sigma_x dz_{x_i} \tag{15}$$

where continuous rates of change $d\log y_{ji}$ and $d\log x_{ji}$ stem from a deterministic term, $s_j dt$, which is the same for both components and assumed to be constant throughout the process[1], plus a random term, dz_{y_i} or dz_{x_i}, specific to realisation i.[2] The summation of all dt, t, reflects the length of the accounting period during which the generation of the j^{th} financial statement takes place, typically one year.

By exponentiation, (14) and (15) leads to

$$\frac{dy_{ji}}{y_{ji}} = (s_j + \frac{\sigma_y^2}{2})dt + \sigma_y dz_{y_i}$$

and

$$\frac{dx_{ji}}{x_{ji}} = (s_j + \frac{\sigma_x^2}{2})dt + \sigma_x dz_{x_i}$$

which, after integration, yields

$$y_{ji} = y_0 e^{s_j t} e^{\sigma_y \mathcal{Z}_{y_i}} \qquad \text{and} \qquad x_{ji} = x_0 e^{s_j t} e^{\sigma_x \mathcal{Z}_{x_i}}$$

where y_0, x_0 are arbitrary constant magnitudes and $\mathcal{Z}_{y_j}, \mathcal{Z}_{x_j}$ are Wiener processes. Ratios of variables generated as above,

$$r_{ji} = r_0 e^{\mathcal{Z}_i}, \tag{16}$$

exhibit no systematic drift.[3] The term $\mathcal{Z}_i$ is also a Wiener process, with variance $(\sigma_y^2 + \sigma_x^2 - 2\,\rho\,\sigma_y\,\sigma_x)\,t$, ρ being the correlation coefficient between z_{y_i} and z_{x_i}.

Ratio components y_{ji} and x_{ji} in (16) obey (13), the robust formulation of scale invariance. If the two processes above were assumed to equate effective rates on average, they would be described as

$$\frac{dy_{ji}}{y_{ji}} = r_j dt + \sigma_y dz_{y_i} \tag{17}$$

and

$$\frac{dx_{ji}}{x_{ji}} = r_j dt + \sigma_x dz_{x_i}. \tag{18}$$

This process is similar to (2). It was tested in [5] for the presence of a drift term in the log of the ratio.[4] However, as shown above, continuous processes which obey the robust formulation of scale invariance lead to (16), where the log of the ratio no longer drifts either upwards or downwards.

NOTES

1. The continuous rate of growth $s_j = s\tau_j$ is the same for all items in the j^{th} financial statement, modelling the random effect of size upon financial statement j, irrespective of the variable considered. That is, the driving force behind the relative magnitude of the accounting numbers observed in a financial statement is s_j. For instance, firms which are larger than the industry expectation exhibit positive s_j whereas those which are smaller have negative s_j.

2. Random terms dz_{y_i}, dz_{x_i} are limits of increments of Wiener processes $\mathcal{Z}_{y_i}, \mathcal{Z}_{x_i}$ as the time interval approaches the infinitesimal dt. As mentioned, $dz_{y_i} = z_{y_i}\sqrt{dt}$ and $dz_{x_i} = z_{x_i}\sqrt{dt}$ where z_{y_i} and z_{x_i} are time-independent standard Normal random variables.

3. In (16), the time dependence in the variance of z generates time dependence in the expected ratio because, as the magnitude of z increases, negative and positive realisations are differently treated by the exponentiation, spanning the intervals $\{0, 1\}$ and $\{1, \infty\}$ respectively. This asymmetry, however, is no longer specific to ratios and a simple logarithmic transformation removes it.

4. The distinction between the two processes (14), (15) and (17), (18) is seldom drawn in Financial Economics because, in general, such a distinction would be irrelevant given the context. Nevertheless, when the issue of interest is the existence or not of processes able to remove a common effect such as firm size, the distinction is essential.

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