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Proportionate Growth and the Theoretical Foundations of Financial Ratios

The article proposes a theoretical framework for understanding financial ratios, showing that the multiplicative character of the financial variables from which financial ratios are constructed is a necessary condition of valid ratio usage, not just an assumption supported by evidence. Also, by assuming that firm size is a measurable statistical effect, the article offers an informed reappraisal of the limitations of financial ratios, particularly the well-known limitation of proportionality. The article is divided into two parts, one where ratio components are viewed as deterministic variables and the other where they are random. Such an approach allows the characteristics of ratios to be more easily understood before generalizing the relationship between ratio components to encompass randomness. In the second part, when variability introduced by firm size is treated as a random effect, it is shown that if the accounting variables Y and X used to calculate a financial ratio Y/X are exponential Brownian motion, and if continuous growth rates are equal and proportionate to firm size, this may lead to ratios which are asymmetric but which do not necessarily drift.

Key words: Accounting; Ratio analysis; Theory.

Accounting academics have often considered the widespread use of ratios in financial analysis as somewhat intriguing. Indeed, in one of the earliest contributions to this topic in the accounting research literature, Horrigan (1965) remarked that financial ratios were referred to in textbooks in almost apologetic tones as though their expected utility was extremely low. Horrigan's response to this situation, however, was to seek to dissipate such doubts by describing the statistical characteristics of some widely used ratios in order to demonstrate that they may be useful after all.

Following Horrigan's optimistic review, subsequent empirical research revealed some promising applications of financial ratio analysis (e.g., Beaver, 1966; Altman, 1968). A few years later, however, the scepticism returned, with authors such as Deakin (1976) noticing that the empirical frequency distributions of financial ratios appear to vary widely and, as a result, questioning the validity of analytical methods that assume the normality of ratio data. This prompted Frecka and Hopwood (1983) and others to propose *ad hoc* techniques such as transformation and the trimming or winsorizing of outliers to deal with the unduly influential values

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present in samples of financial ratios, techniques which reflected the apparently widespread belief that there are no general rules underpinning the ratio method.

Adding to the growing doubt concerning the validity of financial ratio analysis, Lev and Sunder (1979) raised some fundamental questions as to whether the use of ratios is motivated by well-founded considerations or whether, in contrast, it is merely a routine. They claimed that almost all of the assumptions required for valid ratio analysis are likely to be violated in practice. In a more specific critique, Whittington (1980) discussed in detail how ratios may not be up to the task. Both Lev and Sunder (1979) and Whittington (1980) stressed that valid measurement using ratios requires proportionality between the components (i.e., $Y = bX$). Since such an assumption seems to be too restrictive, these authors advocated a two parameter regression model ($Y = a + bX$), or similar functional form, rather than the single parameter ratio model. Barnes (1982) went further, suggesting that nonproportionality is also the source of the excessive skewness often found in the frequency distributions of ratios, and that the use of regressions instead of ratios should eliminate both the problem of nonproportionality and that of skewness in distributions. The prevailing scepticism about the usefulness of ratios deepened further when Tippett (1990) claimed that ratios used as norms or benchmarks for interfirm comparison are intrinsically unstable, drifting upwards or downwards over time.

A striking feature of the above-mentioned research is the small impact it has had, both on the practice of financial analysis and in the way empirical research is carried out. One reason for this may be that researchers presuppose that accounting data used in the computation of financial ratios is a special case, too complex for simple, unifying explanations. As a consequence, empirical research concerning financial ratios lacks the level of definition that is required to make the specific assumptions that are needed to draw appropriate inferences. For example, in spite of the insistence that financial ratios fail to remove the effect of firm size from the financial measurement, to date the literature on ratio analysis has not produced a definition of 'size' itself.

In an earlier attempt to base ratio models on broader assumptions, it was demonstrated that a fuller understanding of financial ratios can be achieved by taking account of the behaviour of the two variables from which a financial ratio is constructed, particularly where firm size plays an important part, and some limiting case theoretical ratio models that allow for exponential growth in accounting variables were identified (McLeay, 1986a, 1986b). Further attempts at normalizing observed ratios through transformation led to the conclusion that better defined ratio models are required (Ezzamel and Mar-Molinero, 1990). Subsequently, Trigueiros (1995) offered a simple explanation for the diversity of distributions found in ratios. While providing empirical evidence that the accounting variables used to calculate financial ratios follow a multiplicative rather than an additive law of probabilities, Trigueiros pointed out that such behaviour suggests the existence of a statistical effect that is common to all of the figures reported in a particular set of accounts but which varies both through time and across firms.

In this article, it is shown that the multiplicative behaviour of ratio components is not just a reasonable assumption supported by empirical observation, but is a prerequisite of valid ratio usage. Furthermore, by also assuming that firm size is a measurable statistical effect, the paper is able to offer a more focused discussion of the limitations of financial ratios, thus adding to our understanding of financial ratios and their potential use.

THE VALIDITY OF RATIOS

Financial analysis is just one of many tasks where ratios are used. There are numerous applications where the usefulness of ratios is evident, such as maps and scaled models that are governed by a ratio or scale that measures the number of times the representation is smaller than reality.[1] Many of the ratios used in financial analysis are similar to scales: interest cover, for instance, is the number of times earnings is greater than interest; the liquidity ratio is the number of times liquid assets exceeds current liabilities; the sales margin expresses earnings, the numerator, as a fraction of sales, the denominator; and so on.[2]

The similarity with scales may also help to understand why financial ratios may be invalid. Scales are arbitrary proportions, chosen before the actual drawing of a scaled representation such as a map takes place. As such, they are neither valid nor invalid. Financial ratios, by contrast, are supposed to represent some pre-existing reality, namely the natural relationship between the numerator and the denominator of the ratio. To the extent that ratios are models of an underlying reality, they may be invalid if the natural relationship between the numerator and the denominator, although pre-supposed to be a simple scale, actually cannot be so.

The traditional approach is to question the validity of financial ratios on these grounds, that is, that a scaling factor between the numerator and the denominator

[1] Speed, flow and pressure can also be considered as ratios, measuring the average change in one variable (the numerator of the ratio) per unit change in another variable (the denominator). A derivative, that is, a rate of change, is also a ratio, in this case a ratio of two small changes. Logarithms also perform a task similar to that of ratios, as measurement on a logarithmic scale is a rate of change. For example, in logarithms, a change from 1 million to 2 million is the same as a change from 1 billion to 2 billion, the logarithmic rate of change expressing the fact that, in both cases, the second observation is twice as large as the first. Unfortunately, an influential fallacy concerning logarithms was introduced by Eisenbeis (1977) in his oft-cited exposition of certain pitfalls in discriminant ratio analysis: 'log-transformed variables give less weight to equal percentage changes in a variable where the values are large than when they are smaller . . . the implications would be that one does not believe that there is as much difference between a \$1 billion and a \$2 billion size firms as there is between a \$1 million and a \$2 million size firms. The percentage difference in the log will be greater in the latter than in the former case' (p. 877).

[2] As with scales, the components of many financial ratios are measured in the same unit, money in this case. In the ratio, as in the scale, the original units of measurement are no longer present. A sales margin of 16 per cent, for example, is a percentage with no units attached. Where the unit in which the numerator of a ratio is measured differs from that of the denominator, ratios may retain both units as in the case of earnings per share, or assume another, as in the case of number of days sales in debtors.

of the ratio may not exist. Unfortunately, rather than investigating just the existence or not of scaling factors, most of the extant literature has adopted the more stringent stance of evaluating models in which the scaling factor is also an *expected* ratio. This additional requirement is unduly restrictive, and is implicit for example when financial ratios are described using statistical models of the additive type.[3] In order to circumvent the drawbacks associated with the *a priori* use of particular models, this section first derives the natural form of proportionality, from which it is possible to obtain a better understanding of the scaling factor. Then, the nature of firm size is discussed and it is shown how size is removed from the financial measurement. Finally, it is also shown that ratios are valid only where size evolves exponentially.

Scale Invariance

When a ratio is used for the purposes of scaling, the implicit assumption is that the relationship between the numerator and the denominator should remain constant no matter what changes are observed in the ratio components, that is,

$$\frac{Y}{X} = \text{Constant.}$$

For instance, the once popular benchmark of 2:1 for the current ratio is deemed to hold whether the figures involved for current assets and current liabilities are as large as \$2bn:\$1bn or as small as \$200:\$100. It is this that prompted authors such as Lev and Sunder (1979) and Whittington (1980) to state that the most basic requirement of ratio validity is proportionality between Y and X and to examine the two characteristics of the proportional relationship, namely linearity and a zero intercept.[4]

As we show below, however, these characteristics do not provide the most general description of proportionality. In particular, it may be noticed that the above emphasizes the relationship between Y and X but not the way in which changes in Y should relate to changes in X so that the ratio remains constant. Yet a thorough understanding of ratios requires this dynamic approach. It is obvious that, when Y is proportional to X, the rate of change of Y with respect to X remains constant and similar to the ratio itself. For ratios to be valid, therefore, the following relationship must hold:

$$\frac{Y}{X} = \frac{dY}{dX}$$

[3] Additive statistical models, which are by far the most frequently used models, add together effects such as the expectation, co-variances and a random term that is assumed to be normally distributed.

[4] It is worth emphasizing that our concern is with ratios, where scaling takes place. Some financial indicators can be viewed as fundamental magnitudes in their own right. For instance, return on assets may proxy the internal rate of return. In such cases an expectation may be modelled in a manner other than the financial ratio, the issue of the validity of the ratio method not applying in such circumstances.

where dY, dX are any related changes observed in Y and X. This is the differential equivalent to Y/X = Constant. It fully encompasses the traditional definition of ratio validity and has the important advantage of making explicit the general requirement of proportionality. In fact, by rearranging terms, the above becomes

$$\frac{dY}{Y} = \frac{dX}{X} \tag{1}$$

Equality (1) shows that the general condition of proportionality is *scale invariance*, whereby the relative change in Y should be equal to the relative change in X. It will be seen below that, besides scale invariance, the validity of ratios also requires that each of the variables Y and X follows a *multiplicative process*, that is, one of exponential change. From these new conditions, which are discussed in greater detail below, it is possible to infer the role of firm size in financial ratios.

Scale Invariance and Firm Size

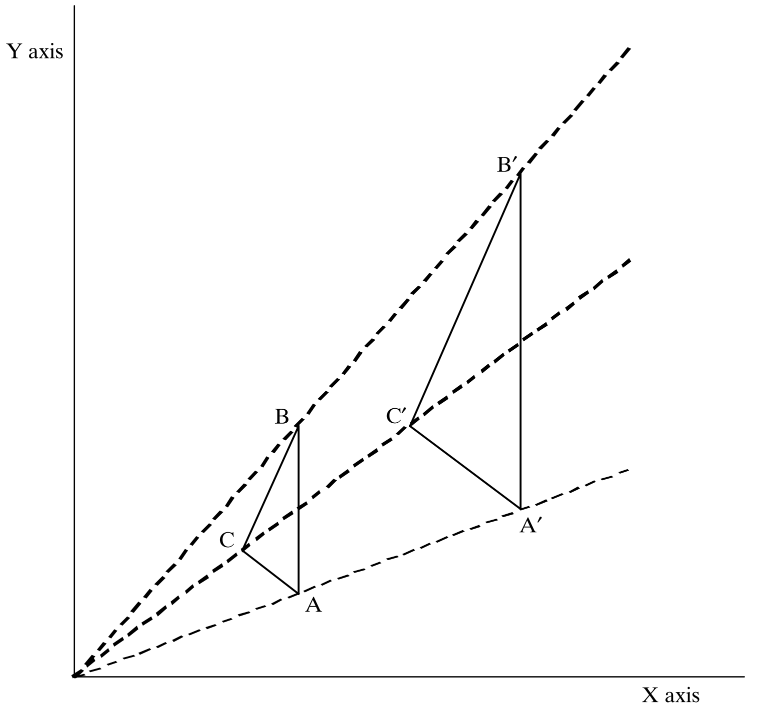
As mentioned above, (1) shows that ratios are valid only if the relative changes in the numerator and denominator are expected to be similar. For instance, when comparing firms in cross-section, if the current assets figure is expected to be many times larger in one firm than in another then this should also apply to current liabilities or any other variable that is potentially useful as a component of a ratio. Similarly, in a time series analysis, (1) implies that variables eligible as the components of a ratio are predicted to grow at the same rate. If, say, sales is predicted to grow by 12 per cent in a given year, then scale invariance requires that earnings should also grow by 12 per cent during the year. It is important to note that (1) describes a mechanism, not observed values. So long as the mechanism underlying reported numbers predicts that dY/Y is similar to dX/X, proportionality is verified irrespective of the actual changes in Y and X. Indeed, without presupposing scale invariance, it would be impossible to arrive at any reliable conclusion about the financial characteristics of firms using ratio analysis. For example, unless it is postulated that relative changes are the same for current assets and current liabilities across firms of different sizes, it would not be possible to infer whether a current ratio above the norm in a large firm is attributable to the liquidity of that particular firm, or is just a characteristic of large firms.

It should not be surprising that the validity of ratios is conditional on the equality of relative changes in variables. Since ratios are scales, they are valid only where the scaling of the data makes sense and this implies *scale invariance* as a property of such data.

Figure 1 provides an intuitive description of scale invariance, where the triangle ABC is scaled up by a factor of 2. Scale invariant changes such as those leading from ABC to its image $A'B'C'$ require that vectors $[X_A, Y_A]$, $[X_B, Y_B]$ and $[X_C, Y_C]$ obey the same law of growth, a law generating changes of equal proportion in each vector as in equality (1). Similarly, scale invariance implies that each of the different items reported in a set of accounts will be influenced jointly by the same effect, and firm size is the most obvious such effect that all of the items reported

FIGURE 1

AN EXAMPLE OF SCALE-INVARIANCE (ABC : A′B′C′)



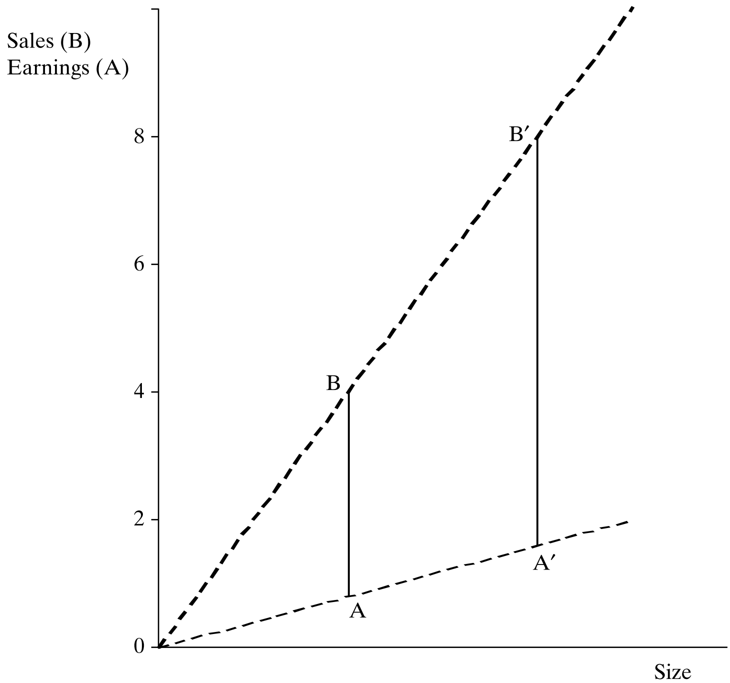
by a firm at a particular point in time will have in common.[5] Thus, it is reasonable to suppose that relative changes in these variables will reflect, *inter alia*, differences in the size of firms or, in a time series context, the growth rate of the firm.

Figure 2 shows how differences in firm size relate to the magnitude of ratio components under the condition of scale invariance illustrated in the previous figure. Suppose that the X-axis now measures firm size and the Y-axis is allowed to represent sales and earnings. Each accounting variable is now modelled as a specific *scale to size* that is characteristic of the variable (which, in this example, is $Y_B/X_B = Y_{B'}/X_{B'} = 4$ for sales and $Y_A/X_A = Y_{A'}/X_{A'} = 0.2$ for earnings), multiplied by the common effect of size. That is,

[5] Not only is it an empirical fact that the financial statements of large firms contain reported numbers that are many orders of magnitude larger than those in the accounts of small firms, there are also compelling economic reasons to support the conviction that each firm's actual size greatly influences the overall magnitude of numbers reported in its accounts. Indeed, if variables such as sales or earnings were not closely related to size, then profitability and dividend yield would be diluted by any increase in size and firms would carefully avoid growing.

FIGURE 2

SCALE TO SIZE FOR EARNINGS (AA′) AND SALES (BB′)



$$\text{Earnings} = 0.2 \times \text{Size} \quad \text{and} \quad \text{Sales} = 4 \times \text{Size}$$

The natural scale between earnings and sales (5 per cent) is given by the division of the two scales to size involved, that is, 0.2 ÷ 4. This value of 5 per cent is thus the ratio norm, which is used to benchmark comparisons across firms or through time. It is because the same size multiplier is present in both components of the ratio, although differently scaled, that removal of the size effect may take place.

It may be concluded that proportionality leads directly to the modelling of accounting variables used in ratios, not as an addition of effects, but as the product of a constant scale by a size factor. Moreover, proportionality is verified no matter the actual behaviour of this size factor. This suggests the need to consider another condition of valid ratio analysis, not just proportionality but also the way in which firm size changes.

Multiplicative Processes

The way a *change* in a variable relates to its *level* is an important consideration. This is because any increase or decrease must occur in such a way that relative changes in the variable remain constant, otherwise such changes would be size dependent and this would render ratios incomparable across firms of different sizes. Specifically, for a given financial indicator, the likelihood of observing a discrepancy of, say, 1 per cent in relation to the norm, must be the same for all firms, whether small or large. A second condition of valid ratio analysis is therefore homoscedasticity: the distribution of the ratio must be independent of size.

In order to illustrate the practical consequences of this new condition, it is necessary to model not just changes in accounting variables but their joint evolution under the influence of firm size. To this effect, a suitable variable τ is now introduced, and the mechanism that drives relative changes in Y and X is written as

$$\frac{dY}{Y} = s_y d\tau \quad \text{and} \quad \frac{dX}{X} = s_x d\tau$$

where s_y, s_x must be strictly independent of Y, X respectively so that the second condition, that of homoscedasticity, is satisfied. This is better known as the 'Law of Proportionate Effect' or 'Gibrat's Law', the source of the family of multiplicative processes mentioned earlier where relative changes observed in Y and X must be homogeneous (Gibrat, [1931] 1957). Variables evolving exponentially with τ, such as

$$Y = Y_0 e^{s_y \tau} \quad \text{and} \quad X = X_0 e^{s_x \tau} \tag{2}$$

are the simplest instance, Y_0, X_0 being arbitrary constant magnitudes in this case. Furthermore, the condition of scale invariance is satisified where $s_y = s_x$.[6]

Table 1 compares the case where size is an exponential function of τ with the alternative where changes in size are linear. Once again, sales is four times size and earnings is one-fifth of size, the natural scale between earnings and sales being 5 per cent. The table shows how ratios that are proportional may nevertheless exhibit relative changes in Y and X that are correlated with size. When changes in size are linear (see Panel 1), the smallest firm illustrated maintains its level of profitability whilst increasing its sales and earnings by 1 per cent, whereas this increase is only 0.75 per cent in the case of the largest firm. On the other hand, any firm keeping its sales margin at a constant level of 5 per cent whilst sales and earnings grow steadily by 1 per cent (see Panel 2) would, *ipso facto*, exhibit exponential growth not linear growth. Whilst in both cases the relationship between the numerator and the denominator of the ratio is exactly the same straight line with a zero intercept, it is only when the changes in Y and X are governed by a multiplicative process that the ratio may be interpreted independently of firm size.

Whereas the first of the conditions introduced in this paper (scale invariance) demands a specific type of relationship between components, that is, that relative changes in the ratio components will be the same for the numerator Y and

6 Notice that whereas (1) equates the two effective rates of change dY/Y and dX/X, (3) is based on $e^{s_y \tau}$ and $e^{s_x \tau}$ which are continuous rates. The relationship between an effective rate of change, r, and the underlying continuous rate of change s is $s = \log(r + 1)$.

TABLE 1

LINEAR AND EXPONENTIAL GROWTH IN RATIO COMPONENTS

Panel 1: Linear changes in size (the ratio *Y/X* is a constant 5% but the relative changes in *X* and *Y* are correlated with size)

τ	Size	Earnings (Y) $= 0.2 \times$ Size	Sales (X) $= 4 \times$ Size	Y/X	dY/Y	dX/X
0	100	20.0	400	5%		
1	101	20.2	404	5%	0.0100	0.0100
2	102	20.4	408	5%	0.0099	0.0099
. . .						
32	132	26.4	528	5%	0.0076	0.0076
33	133	26.6	532	5%	0.0075	0.0075

Panel 2: Exponential changes in size (the ratio *Y/X* is a constant 5% but the relative changes in *X* and *Y* are independent of size)

τ	Size	Earnings (Y) $= 0.2 \times$ Size	Sales (X) $= 4 \times$ Size	Y/X	dY/Y	dX/X
0	100.00	20.000	400.00	5%		
1	101.00	20.201	404.02	5%	0.0100	0.0100
2	102.02	20.404	408.08	5%	0.0100	0.0100
. . .						
32	137.71	27.543	550.85	5%	0.0100	0.0100
33	139.09	27.819	556.38	5%	0.0100	0.0100

denominator X, the second condition (the law of proportionate effect) is a constraint that is placed separately on the individual ratio components whereby a relative change in Y will be independent of the actual value of Y and a relative change in X will be independent of the actual value of X. Since ratios are used in two clearly distinct modes of analysis (time series and cross-section), the interpretation of (2) will vary accordingly. Its application in these different contexts (i.e., for either trend analysis or interfirm comparison) is illustrated below.

A Time Series Example

In a time series context, the interpretation of processes such as (2) is straightforward, τ representing a time sequence starting when $Y = Y_0$, $X = X_0$. These initial values should be viewed as reflecting the initial size of the firm, each variable being scaled differently to size. Furthermore, s_y, s_x are real growth rates observed in Y, X. Although these growth rates may vary with time, by fluctuating randomly around a constant mean for example, they are expected at any one moment in time to be equal for different variables (i.e., $s_y = s_x = s$) since they reflect proportionate changes in the size of the firm.

Deviations observed in s_y or s_x in relation to the growth rate s may be expressed as a difference $n_y = s - s_y$ or $n_x = s - s_x$. Suppose that, in a given year, the sales figure is reported as \$1,000 and earnings as \$100, then profitability is 10 per cent. If sales grows by s (the same as the firm) but earnings grows by only $s - n$, then, in the following year,

$$\text{Earnings} = 100e^{(s-n)\tau} = 100e^{s\tau}e^{-n\tau} \quad \text{and} \quad \text{Sales} = 1000e^{s\tau}$$

and the ratio decreases by $e^{-n\tau}$. Not only is the ratio insensitive to the magnitudes of sales and earnings in the previous year, it is also insensitive to the growth rate s. The second condition of the valid ratio method is thus satisfied since the sales margin remains comparable, however low or high the growth of the firm may be.[7]

A Cross-Sectional Example

In cross-sectional analysis, an intuitive meaning may also be given to s and τ. In this case, τ measures the dispersion of size. Industries where firms range from the very small to the very large exhibit high τ whereas those where size is homogeneous exhibit low τ. Accordingly, s_y and s_x are the distance between the realization Y or X and the corresponding median Y_0 or X_0. As in the previous example, observed s_y and s_x may differ but they should be viewed as realizations of a common, underlying s so that the condition of scale invariance is satisfied.[8]

The ratio of the medians Y_0/X_0 is the natural scaling factor between Y and X. If, for example, the median sales in a given industry is \$1,000 and for earnings it is \$100, whereas for firm A both sales and earnings are s standard deviations from the norm, then

$$\text{Earnings}_A = 100e^{s\tau} \quad \text{and} \quad \text{Sales}_A = 1000e^{s\tau}.$$

Now consider firm B which is more profitable, the same volume of sales generating earnings n standard deviations above the norm, that is,

$$\text{Earnings}_B = 100e^{(s+n)\tau} = \text{Earnings}_A e^{n\tau} \quad \text{and} \quad \text{Sales}_B = \text{Sales}_A.$$

The difference in profitability between B and A, which is shown above to be $e^{n\tau}$, is independent of s.[9] With multiplicative processes such as (2) governing accounting variables, ratios are comparable across firms however small or large they may

[7] Under sustained conditions, the ratio decreases by e^{-n} over one year, by e^{-2n} over two years, and so on. Since this effect accrues over time, comparisons using periods of different lengths are possible.

[8] In fact, the role of τ is similar in both cross-section and time-series analysis, allowing for comparison between cases where the time period or the dispersion are not the same.

[9] The increment in the ratio, $e^{n\tau}$, is also independent of the industry norms, Y_0, X_0. Indeed, from the practical viewpoint, independence from industry norms, initial values, targets, benchmarks and so on should be regarded as an important condition of ratio validity, as practitioners make decisions based on measurement of deviation from these norms.

be.[10] Not only is the ratio independent from the magnitudes of its components, differences in profitability are also size independent as required by the second condition.

Negative Values

A further issue that should be addressed at this point concerns the way in which negative values should be treated. In the case discussed earlier, where two multiplicative variables Y and X form the ratio Y/X, the ratio itself will take positive values only. This is the situation with the current ratio, for instance. Yet it is obvious that certain other ratios, such as return on sales, permit negative as well as positive values.

The problem of negative values in ratios may, in general, be avoided by transforming Y/X. Indeed, as most accounting variables are governed by relatively simple accounting identities, many financial ratios can be expressed by rearranging Y and X. For instance, as earnings may be represented as $X - Y$, where Y represents sales and X represents total costs, then the ratio r (return on sales, in this case) is equivalent to $(X - Y)/X$, or $1 - Y/X$. Similarly, where total assets is identical to the sum of equity capital and total liabilities, the liabilities ratios would be defined as $r = X/(Y + X)$, or $(1 + Y/X)^{-1}$. Table 2 shows how the meaningful rearrangement of Y and X leads to a set of ratios encompassing the classes most commonly observed in practice, each a function of Y/X.

Some Limitations

It has been argued above that linearity and a zero intercept are not sufficient to ensure valid ratios, but that a scale invariant, homogeneous process governing changes in ratio components is also required. In this context, it would seem that the most obvious characteristics of accounting data that may limit the use of ratios as valid measures of financial interrelationships are:

The presence of a constant term, such as fixed costs, which creates nonproportionality between the components of ratios, thus invalidating (1). This is a well-known limitation, generally presented as the main challenge to the valid use of financial ratios. In Figure 3, the proportionality between earnings and sales shown in Plot I can be compared with the effect of deducting a small constant term from earnings in Plot II. This is a local distortion having a significant effect only on those values in the region of the constant.

The presence of factors which prevent accounting variables from changing at similar rates, thus leading to ratio nonlinearities. Plot III of Figure 3 shows the effect of

[10] Note that nonmultiplicative processes lead not only to size-dependent but also to norm-dependent ratios. Where realizations are proportional to size (e.g., Earnings = $4s\tau$ and Sales = $40s\tau$), such processes will satisfy the two 'traditional' conditions of ratio validity, linearity and a zero intercept, as shown in Table 1. Moreover, they will also obey (1), at least if τ is not allowed to approach zero. However, deviations in the ratio will be influenced not only by the size $s\tau$ but also by the natural scale of 10 per cent (i.e., $4s\tau/40s\tau$), a gain $d\tau = n$ in earnings, for instance, increasing the ratio by 10 per cent $n/s\tau$.

TABLE 2

CLASSES OF FINANCIAL RATIO

Ratios	Transformation	Boundaries	Examples	
$\frac{Y}{X}$	r	$0, \infty$	Current ratio	$\frac{\text{Current assets}}{\text{Current liabilities}}$
$\frac{X - Y}{X}$	$1 - r$	$-\infty, 1$	Sales margin	$\frac{\text{Sales} - \text{Costs}}{\text{Sales}}$
$\frac{Y - X}{X}$	$r - 1$	$-1, \infty$	Change in capital employed	$\frac{\text{Closing capital} - \text{Opening capital}}{\text{Opening capital}}$
$\frac{Y + X}{X}$	$1 + r$	$1, \infty$	Interest cover	$\frac{\text{Earnings} + \text{Interest paid}}{\text{Earnings}}$
$\frac{X}{Y + X}$	$\frac{1}{r} - 1$	$0, 1$	Liabilities ratio	$\frac{\text{Liabilities}}{\text{Equity} + \text{Liabilities}}$
$\frac{X}{Y - X}$	$1 + \frac{1}{r}$	$0, \infty$	Financial leverage ratio	$\frac{\text{Liabilities}}{\text{Total capital} - \text{Liabilities}}$

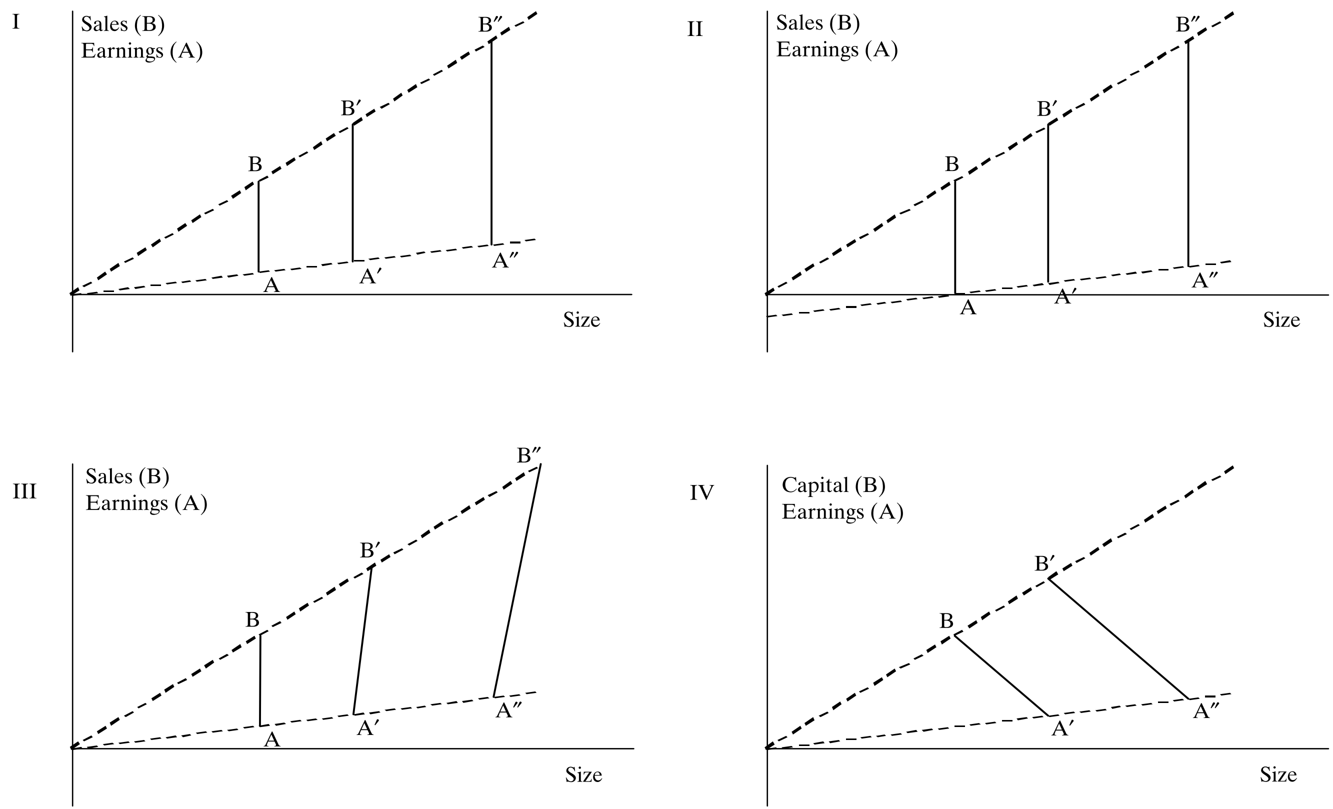
factors that have a differential impact on the two components of a ratio. In this example, sales now grows faster than earnings—or, in the cross-sectional context, increases disproportionately from one firm to another in the face of the diseconomies of scale discussed in Whittington (1980)—and the ratio decreases with size. This is potentially far more serious than the effect of a constant term. Even small differences between rates of change can introduce into the ratio an exponential correlation with size.

The use of lagged ratio components, where the numerator and denominator are not taken from the accounts for the same period and, therefore, where the effect of size is not removed because the same measure of size is not present in both components of the ratio. Specifically, each case will exhibit a displacement relative to the natural scale and proportional to the firm's actual rate of growth. Plot IV of Figure 3 depicts the influence of a lagged denominator, where the ratio of earnings to capital (a ratio for which it is usual to take the denominator from the previous year's financial report) is stable provided that growth is also stable, with the ratio suffering a constant displacement in proportion to such growth.

Figure 3 provides yet another reason for considering that linearity and a zero intercept are not sufficient conditions for valid ratios. Where, given these two conditions, the chords AB, $A'B'$ and $A''B''$ are parallel with each other, then the ratio is constant. However, as shown by comparing Plot IV with Plot I, this does not necessarily remove the effect of size. Only where the chords are perpendicular to the X-axis are the ratio components scaled to the same measure of size, and only then can the ratio measurement be described as size independent.

FIGURE 3

I: SCALE-INVARIANCE; II: ADDITIVE TERMS; III: DIFFERENCES IN PROPORTIONATE GROWTH; AND IV: LAGGED DENOMINATORS



RATIOS OF RANDOM VARIABLES

When considering the statistical properties of financial ratios, most authors have assumed that the difficulties posed by their atypical behaviour are caused by distortions of normality. However, financial ratios cannot be described as resulting from the kind of additive stochastic process that underlies normal variables. Indeed, the same can be said for the accounting variables from which financial ratios are constructed, given the nature of the double-entry bookkeeping system that is based on aggregation of like transactions. As with many other economic phenomena such as wealth and the size of firms, the amount that is reported for a financial statement item is generated by a multiplicative rather than an additive law of probabilities. That is, whilst each transaction adding to the amount reported as, say, total sales can be modelled as a random event, the transaction contributes to the reported aggregate not in a manner that could lead to either an increase or decrease in total sales, but by accumulation only. Such accumulations of random events are described as multiplicative, as opposed to additive, because their likelihood is conditional on the occurrence of a chain of prior events, and stems therefore from the product rather than the sum of the individual probabilities.[11]

Characteristics of Multiplicative Variables

Multiplicative variables tend to be lognormally rather than normally distributed. Thus, they cannot be treated as distortions of normality. No distorting mechanism would be able to create, from additive events, the wide range of values generally found in multiplicative variables. For instance, the larger values observed in a lognormal sample are likely to be many times greater than the median. Such extreme proportions have no counterpart in additive variables where the likelihood of observations two or three standard deviations above or below the mean is very small.[12]

When the multiplicative character of accounting data is ignored, features of the data that would otherwise be considered as commonplace (such as positive skewness and extreme values) are likely to be seen as extraordinary. In fact, the presence of so-called outliers in empirical ratio frequencies is most likely to be a

[11] The same issues arise in modelling count data, where the transition from one state to the next may be described as a multiplicative process. Statistical models based on counting processes, including a number of applications of the multiplicative intensity model, are discussed in Andersen *et al.* (1993).

[12] Details concerning the density of the bivariate lognormal distribution can be found in Johnson (1987). Lognormality in variables such as sales, earnings and assets, received a great deal of attention in texts on the theory of the growth of firms. In the accounting literature, McLeay (1986a) assumes lognormality in variables that are sums of similar transactions with the same sign such as stocks, creditors or current assets and McLeay (1997) considers this further, showing that two lognormal variables forming the ratio $X/(Y + X)$ have a logistic distribution which is found to give a good fit to observed frequencies of financial ratios of this form. Other empirical evidence (Trigueiros, 1995; Sudarsanam and Taffler, 1995) suggests that lognormality of the accounting variables used in calculating financial ratios is widespread.

consequence of multiplicative skewness in ratio components.[13] Given that such observations are not unusual in financial ratio samples, the technique of trimming or outlier deletion advocated by authors such as Frecka and Hopwood (1983) is a questionable practice. In fact, since the mechanism commanding the emergence of extreme values holds when the scale differs, and as trimming is in some respects equivalent to a reduction in scale, it will be found that the exclusion of one extreme value merely leads to its replacement by another extreme value on the reduced scale.[14]

The Random Effect of Size

A further reason why it is important to assume the correct type of statistical behaviour is that adequate descriptions of the interrelationships amongst variables differ between additive and multiplicative variables. For additive data, it is assumed implicitly that distributions are preserved when variables are added or subtracted. This is not the case for multiplicative data where distributions are preserved when variables are multiplied or divided. For example, the simplest additive formulation is $x = \mu + z$, where x is equal to an expected value, μ, plus a random deviation, z. The multiplicative equivalent would be $x = x_0 f$, where a realization of x is explained by the pre-specified level, x_0, multiplied by a random factor f specific to each case.

When an additive variable x is explained not only by an expected value, μ, but also by δ_j, an extra component of the variance of x, then $x_j = \mu + \delta_j + z$, where δ_j is the expected deviation from μ introduced by the j^{th} level of δ. If the same δ_j is present in two variables, y and x, it is possible to remove it from measurement by subtracting variables. For instance, when a medical trial is carried out in the same group of patients both before and after treatment, the difference between observations y and x measures the effect of the treatment and is free from spurious influences such as those of the sex of the patient because such factors, being present in both observations, cancel out when subtracted. The ratio method may be viewed as the multiplicative equivalent of the above example, where firm size is the 'spurious' influence to be removed from the ratio components y, x. In fact, the simplest generalization of the deterministic processes in (2) to allow for randomness would lead to the definition of accounting variables y, x, as reported in the j^{th} financial statement, as

$$y_j = y_0 e^{s\tau_j} f_y \quad \text{and} \quad x_j = x_0 e^{s\tau_j} f_x \tag{3}$$

13 Not all of the outliers found in ratios are caused by multiplicative skewness. After removing such skewness (for example, by using the logarithmic transformation), the distribution of ratios often turns out to be leptokurtic. Fat tails also create outliers, though on a smaller scale.

14 A regression where sales explains earnings using data from the U.K. electronics industry can illustrate this. When the Cook Distance (Cook, 1977) is used to identify influential cases for 1988, for example, two firms (GE and STC) are singled out as outliers. After trimming these two firms, three new firms (Sunleigh, English Electric and Brother International) become influential. After also excluding these three firms, one more firm (Synapse Computers) emerges with a new Cook Distance of 80, a value that indicates extreme influence.

y_0, x_0 are arbitrary constant levels of y, x and the common term $e^{s\tau_j}$ plays the same role as δ_j above: it is the component of the variance of y, x present in all observations in the j^{th} financial statement. f_y, f_x are random proportions of y, x unexplained by the model.[15]

In the above *random effects* models,[16] y_0, x_0 are specific to variables y, x (they are independent of the particular financial statement under analysis), whereas $e^{s\tau_j}$ (the extent to which any item in the j^{th} financial statement is likely to be larger or smaller than the constant level) scales all items in the financial statement irrespective of changes in the particular variables of interest. Similarly to (2), this separation between effects ensures that the ratio y/x removes $e^{s\tau_j}$, the effect attributable to the j^{th} financial statement, thus making company accounts comparable either through time or across firms.

Stochastic Scale Invariance

It may be asked whether the conditions of ratio validity leading to (1) are feasible when, as in the above model, variables are random. Scale invariance as defined by equality (1) applies to deterministic changes where the rules of classical differential calculus apply, but not to continuous time processes where stochastic calculus would apply instead of differential calculus.[17] However, it should not be concluded that it is impossible to observe scale invariance in the relationship between stochastic variables. The reason why (1) fails to encompass such variables is that it equates two *effective* rates of change whereas an equality between two *continuous* rates is now required. A formulation of the scale invariance condition that is robust regarding the nature of the variable (i.e., deterministic or stochastic, continuous or discrete) is obtained by equating expected continuous rates of change, as follows:

$$\mathrm{E}\left[\log\frac{y+dy}{y}\right] = \mathrm{E}\left[\log\frac{x+dx}{x}\right]$$

which may be abridged as

$$d\log y = d\log x. \tag{4}$$

Suppose that the two components of a ratio y and x are generated by the stochastic differential equations

$$d\log y_j = s_j dt + \sigma_y dz_y \quad \text{and} \quad d\log x_j = s_j dt + \sigma_x dz_x \tag{5}$$

15 In cross-sectional analysis, the median provides an appropriate 'constant' reference level, and likewise the initial value in a time-series context.

16 Effects model differences in relation to the expectation, where such differences are introduced by components of the variance. Influences such as the two possible sexes are referred to as 'fixed effects' because the component introduced by the difference in sex is deterministic. Where the component is itself a random variable, the effects are called 'random effects'. Size is therefore a random effect. The different financial statements present in a sample represent the levels of an effect. In statistical terms, the accounting numbers in the same financial statement are said to belong to the same level.

17 A useful introduction to stochastic processes in the context of financial economics is available in Dixit and Pindyck (1994, pp. 59–82) and Campbell *et al.* (1997, pp. 339–49).

where continuously compounding rates of change $d\log y_j$ and $d\log x_j$ stem from a deterministic term $s_j dt$,[18] which is the same for both variables and constant throughout the process thus satisfying the conditions of ratio validity, plus a random term dz_y or dz_x specific to y or x with standard deviation σ_y or σ_x respectively.[19] The summation of all dt, t, reflects the length of the accounting period during which the generation of the j^{th} financial statement takes place, typically one year.

Now, by exponentiation, (5) leads to

$$\frac{dy_j}{y_j} = \left(s_j + \frac{\sigma_y^2}{2}\right)dt + \sigma_y dz_y \quad \text{and} \quad \frac{dx_j}{x_j} = \left(s_j + \frac{\sigma_x^2}{2}\right)dt + \sigma_x dz_x$$

which, after integration, yields the stochastic equivalent to (2) or (3), that is,

$$y_j = y_0 e^{s_j t} e^{\sigma_y z_y} \quad \text{and} \quad x_j = x_0 e^{s_j t} e^{\sigma_x z_x}$$

where y_0, x_0 are arbitrary constant levels, as above, and z_y, z_x are Wiener processes, that is, the time-transformed Brownian motion with independent, normally distributed mean zero increments that is present in Markov chains. Ratios of variables generated as geometric Brownian motions, as above, evolve as

$$\frac{y_j}{x_j} = \frac{y_0}{x_0} e^z \tag{6}$$

thus removing s_j, the effect of size from measurement. The term z is also a Wiener process, with variance $(\sigma_y^2 + \sigma_x^2 - 2\rho_{yx}\sigma_y\sigma_x)t$, ρ being the correlation coefficient between z_y and z_x.

The ratio components y_j and x_j in (6) obey the robust formulation of scale invariance (4), and the ratio does not necessarily drift upwards or downwards as previously thought.[20] Nevertheless, the time dependence in the variance of z

[18] Similarly to (3), the continuous rate of growth $s_j = s\tau_j$ is the same for all items in the j^{th} financial statement, modelling the random effect of size upon financial statement j irrespective of the variable considered. Firms that are larger than the industry norm exhibit positive s_j whereas those that are smaller have negative s_j.

[19] The random terms dz_y, dz_x are limits of increments of Wiener processes as the time interval approaches the infinitesimal dt. For practical purposes, $dz_y = z_y\sqrt{dt}$ and $dz_x = z_x\sqrt{dt}$ where z_y and z_x are standard normal random variables that are time independent. Wiener processes are continuous time Markov processes with independent, normally distributed increments with zero mean, the variance of a Wiener process being proportional to time. Thus, y and x obey the Markov condition whereby only the most recent realization contains information useful to predict future realizations. Markov processes are used extensively as plausible approximations to the way variables such as stock prices or earnings are generated (see Ball and Watts, 1972, pp. 665–6). In the literature on financial ratios, Lev (1969) is an early example of the use of Markov processes.

[20] If the process was assumed to equate effective rates on average, they would be described as

$$\frac{dy_j}{y_j} = \mu_y dt + \sigma_y dz_y \quad \text{and} \quad \frac{dx_j}{x_j} = \mu_x dt + \sigma_x dz_x$$

This is the first and most basic process considered by Tippett (1990), and tested indirectly by Tippett and Whittington (1995) for the presence of a drift term in the ratio, where y_j/x_j is held to be an exponential function of $\mu_y - \mu_x + \sigma_y^2 + \sigma_x^2 - \rho_{yx}\sigma_y\sigma_x$. The present article, however, shows that continuous processes such as (5) which obey the robust formulation of scale invariance lead to (6).

generates time dependence in the ratio because, as the magnitude of z increases, negative and positive realizations are differently treated by the exponentiation, spanning the intervals $\{0,1\}$ and $\{1,\infty\}$ respectively. This asymmetry, however, is no longer specific to ratios and a simple logarithmic transformation removes it.

FINAL COMMENTS

The above discussion may be summarized as follows:

- ratio validity requires not only linearity and an intercept equal to zero in the relationship between the numerator and denominator, but also that proportionate changes in ratio components should be independent of size;
- under such conditions, firm size may be modelled as a common effect specific to each set of financial statements, with each variable being scaled to size and the ratio norm arising from such scaling;
- the statistical properties of certain financial ratios that are thought of as undesirable can be attributed to the aggregation of like transactions under double-entry bookkeeping, but simple transformations can lead to statistically well-behaved ratios;
- mechanisms equating continuously compounding growth rates proportionate to the size of the firm may lead to valid ratios even in the case of random components, and ratios of Markov processes will not necessarily drift.[21]

In setting out these insights into the theoretical foundations of the ratio method, we have addressed the question as to whether the use of financial ratios is motivated more by tradition than by well-founded considerations. It is our conclusion that, in spite of their simplicity, financial ratios are governed by an explicit set of conditions and, moreover, they require data of the type that has been shown to characterize the figures reported in company accounts.

Given these conclusions, one may ask why some previous contributions to the literature on financial ratios have led to such a pessimistic view of ratio analysis. Two reasons may be given. First, it is often assumed that accounting variables are, like many random variables, statistically additive. They are not. The statistical foundations of ratio analysis should be based on the understanding that accounting variables are multiplicative, being governed by proportionate or exponential growth. Second, the way in which firm size influences financial variables has also been misunderstood, leading to much uncertainty as to whether ratios remove size or not.

In this context, we return to the often quoted statement attributed to Lev and Sunder (1979): that almost all of the assumptions required for valid ratio analysis are likely to be violated in practice. The statement is formally correct, of course,

[21] The article also leaves a number of details unexplored. First, other diffusion processes should be tested against the ratio model. Second, the extent to which scale invariance holds for different ratios and the relative importance of size compared to other statistical effects are each in need of clarification. Finally, the extent to which deviations from strict lognormality and a variance which increases with time make the measurement dependent on norms or size is an open question.

but it might as well be applied to Newton's laws of motion and to many other models considered as good enough approximations in normal circumstances. Nevertheless, the statement by Lev and Sunder is misleading. Distortion, in spite of its presence in mathematical models, may be small in specific cases such as in the case of nonproportionality. Moreover, when weighing inaccuracy against the ability to provide an intuitive interpretation with a parsimonious model, the trade-off could prove to be largely favourable to the less accurate methodology.

The kind of trade-off referred to above is particularly relevant to the ratio method. Ratios, having just one degree of freedom, are able to measure deviations from the norm. The condition of constant proportionate growth (or scale invariance) is a direct consequence of this: one unique parameter is able to deal with scale invariant changes, that is, the modelling of the common growth of both components. By providing deviations from the norm, ratios are able to provide in a succinct form the information that financial analysts seek. Conversely, analysts would find it more difficult to use models where the relevant information is contained in several parameter values, and the more complex such models are, the more sensitive they become to irregular data or to influences not necessary in the financial decision process.

Thus, to conclude, the challenge facing research into financial ratio analysis is not how to increase the complexity of models. Rather, it is how to take account of the limitations of the parsimonious ratio model without changing the specific characteristics of the measurement.

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