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PROBING A NOISY OSCILLATOR SYSTEM

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We examine the effect of adding a sinusoidal signal to a system of coupled noisy nonlinear oscillators. When the frequency of this "probe" signal is close to the frequency of the unprobed system we observe a resonance behavior, enabling us to determine the underlying frequency of the noisy system. Furthermore, in the prototype SQUID system we consider here, we find that the frequency of the underlying solution decreases with increasing coupling strength. Combining this finding with the resonance phenomenon we discuss ways to enhance the sensitive of SQUIDs to weak low frequency signals.

Keywords: Noisy oscillators; signal detection; resonance; SQUID.

1. Introduction

Many nonlinear dynamical systems exhibit a bifurcation from a stationary to a time-periodic solution [1]. Finding this bifurcation point, and characterizing the time-periodic solution, is sometimes possible analytically, using techniques that have been developed in the past decades [1, 2]. This, however, typically becomes more difficult when the system consists of a set of coupled elements, each of which can (uncoupled) undergo a bifurcation to a time-periodic solution. In the presence of a noise floor, the problem can become especially complicated; in this case one often no longer has a "sharp" bifurcation, and computing the frequency of the time-periodic solutions can be daunting. In specific dynamical systems, the spontaneous oscillation frequency may be computed for small noise levels, usually via a center-manifold reduction of the dynamics to a 1D normal form [1, 2], or from numerical simulations of such a system [3]. In general, however, a systematic way of finding the frequency is not available.

Knowing the internal frequency of the system can have important implications [4–8]. For example, for the prototype system that we focus on here (the 2 Josephson junction, or dc Superconducting Quantum Interference Device (SQUID)) [9, 10], it has been found that the system reaches its optimal performance for small running frequencies (i.e. close to the bifurcation) [11–13], via a noise-mediated resonance phenomenon that is reminiscent of the stochastic resonance effect [14–16].

In this Letter, we present a technique to find the underlying frequency of a system of coupled noisy nonlinear oscillators which utilizes applying a probing sinusoidal signal [17]. As illustrated in our prototype system, adding the probing signal leads to a clear resonance behavior as the probe frequency approaches the underlying (spontaneous oscillation) frequency of the coupled system. This phenomena has been exploited extensively in spectroscopy to investigate the internal properties of a system [18]. In the coupled SQUID system, this resonant behavior, in the presence of a noise floor arising from thermal noise in the junctions, is demonstrated via Langevin simulations and through the semi-analytic solution of the Fokker–Planck equation describing the dynamics in the mean field limit. We observe that the underlying frequency of the globally coupled system is a function of the coupling strength. In particular, by increasing the coupling strength, we can tune the system closer to the bifurcation and decrease the underlying frequency. This can have important consequences for the ability to detect weak signals with SQUIDs since the sensitivity to external signals is optimal just past the bifurcation. We demonstrate this increased sensitivity for the case of two coupled SQUIDs.

2. The Model and Results

We begin by briefly summarizing the equations governing N coupled dc SQUIDs (for a more detailed background and description see [9, 10, 13]). These equations describe the dynamics via the time-derivatives of the Schrödinger phase differences $\delta_j^{(k)}$ $(j = 1, 2)$ across each Josephson junction of the kth SQUID $(k = 1, \dots, N)$. For simplicity, we assume that all the junctions have identical critical currents I_0 and normal resistances and are identically biased with bias current J (normalized to I_0). We also limit ourselves to the case of global coupling with coupling strength M. This can be accomplished experimentally by shielding each individual SQUID, summing the magnetic fluxes in all SQUIDs using a network of pickup coils and feeding back this average total output flux to each SQUID through a feedback coil. Then, with the appropriate rescalings,

$$\dot{\delta}_j^{(k)} = J + (-1)^j I_k - \sin \delta_j^{(k)} + \xi_k^{(j)}\,, \qquad j = 1, 2\,; \quad k = 1 \dots, N\,, \tag{1}$$

where the screening current I_k in the kth loop can be written as:

$$\beta_k I_k = \delta_1^{(k)} - \delta_2^{(k)} - 2\pi\Phi_0^{-1}\left[\Phi_e + M \sum_{m \neq k} I_m\right]. \tag{2}$$

β_k being the nonlinearity parameter, $\Phi_0 \equiv h/2e$ the flux quantum, Φ_e the external magnetic flux and $\xi_i^{(j)}$ representing junction-generated Gaussian white noises with $\langle\xi_i^{(j)}(t)\rangle = 0$, $\langle\xi_i^{(l)}(t)\xi_j^{(m)}(t')\rangle = 2D\delta_{ij}\delta_{lm}\delta(t-t')$. Since I_m is itself a function of every other screening current, (2) represents an infinite nested series. It is well known [9–12] that a single noiseless SQUID ($\xi = 0$ and $N = 1$) exhibits two regimes of operation: a superconducting state where the phases are time independent and a "running state" where the phases are oscillatory. The latter can be reached by increasing J and keeping Φ_e constant. At $J = J_c$, the stable and unstable fixed points coalesce in a saddle-node bifurcation. Past J_c, two limit cycles exist, one being stable. Since the bifurcation is supercritical the frequency of the oscillatory solutions obeys the characteristic square-root scaling law [1]. For the single SQUID, analytical expressions for both J_c and the running frequency have been recently obtained [19], together with analytic solutions for the dynamics close to the bifurcation [8]; analogous computations have been carried out in a Morris–Lecar model of type-I neurons [20], demonstrating the generality of this description in this class of bifurcations.

For the N coupled SQUID case, the frequency of the running state can also be calculated by applying a center manifold technique close to the bifurcation point. We find that the SQUID with the lowest value of β crosses the bifurcation first, leading to entrainment (but not synchronization) of the other elements. The resulting running frequency for all elements is given by [21], $f = \sqrt{F\alpha - \gamma^2\varepsilon^4}/2\pi$, where F, α and γ are combinations of the system parameter values and $\varepsilon = 2\pi M/\Phi_0$ is a direct measure of the coupling. The most salient feature of this result is that, at least close to the bifurcation, the frequency decreases as the coupling increases. In fact, for large enough values of M the coupling can destroy the running state, bringing the system in the stationary state.

In the presence of noise, the bifurcation point cannot be exactly determined; rather, the system spends, on average, more or less time in the stationary state, depending on the parameters. Calculating the underlying system frequency is also, in general, complicated. Close to the bifurcation point in the deterministic system, a normal form analysis can lead to approximate results. However, this approach is only valid for small noise levels. Thus, for experimentally realistic levels of noise, an alternative method for finding the underlying frequency is needed.

Let us consider now an external flux that has a time-sinusoidal component: $\Phi_e \to \Phi_e + q\sin(\omega_p t)$. We will first examine the effect of this probe signal via direct Langevin simulations. In Fig. 1(a) we have plotted, for two different probe signals, the power spectrum of the average screening current $\bar{I}$:

$$\bar{I} = N^{-1}\sum_{l=1}^{N}\beta_l^{-1}[\delta_1^{(l)} - \delta_2^{(l)} - 2\pi\Phi_0^{-1}\Phi_e]\,. \tag{3}$$

To obtain the powerspectrum, we have averaged 100 timeseries of 2^{23} timesteps each for $N = 2$. Note that the amplitude of the probe signal is smaller than the noise level. Within the noise, the two powerspectra are identical, except at the frequency corresponding to the probe signal, denoted by the symbols. For a probe frequency ω_p far removed from the (running) frequency of the underlying system, corresponding to the broad peak of the powerspectrum, the probe signal is barely discernible (solid square). For a probe frequency with a frequency that matches the broad peak in the power spectrum of the unprobed system the signal is amplified

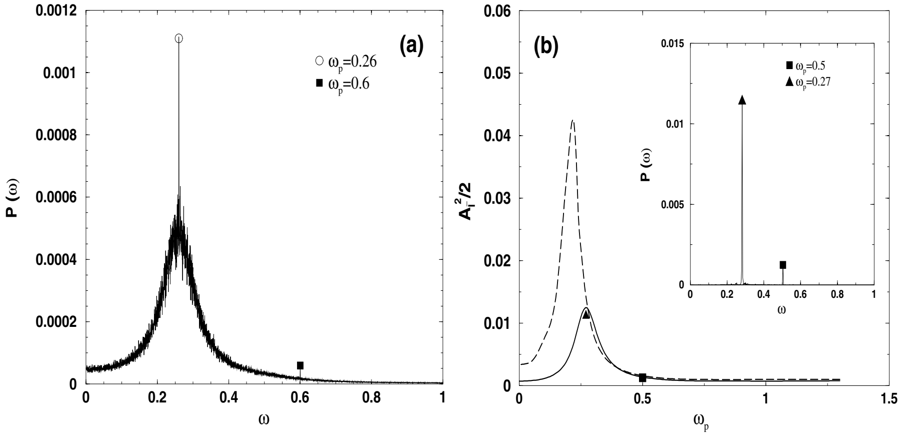

Fig. 1. (a) Powerspectra of the screening current for a probe signal with $\omega_p = 0.26$ and $\omega_p = 0.6$. The peaks at the probe frequency are denoted by a solid square and open circle respectively. Parameter values are $D = 0.05$, $\beta = 1$, $J = 0.4$, $\Phi_e/\Phi_0 = 0.45$, $q = 0.0025$ and $M = 0.02$. (b) $A_{\bar{I}}^2/2$, obtained by solving the NFPE, as a function of the probe frequency for two different values of the coupling strength: $\bar{M} = 0.01$ (solid line) and $\bar{M} = 0.05$ (dashed line). The SQUIDs are identical, with $q = 0.01$ and remaining parameters as in Fig. 2. The inset shows the powerspectra obtained via direct Langevin simulations for $N = 500$ for $\omega_p = 0.5$ (square), and $\omega_p = 0.27$ (triangle). The peaks of these powerspectra are also plotted in the main figure.

dramatically (open circle). Thus, adding a probe signal gives us a tool to investigate the dynamics of the noisy system.

A major drawback of the Langevin approach is its computational cost. This makes simulations for larger number of elements and high values of noise prohibitively expensive. To address the SQUID dynamics for these cases and to allow for a systematic study of the parameter space, we now derive a Fokker–Planck equation (FPE) for the case of $N \to \infty$ globally coupled SQUIDs, by extending our recent derivation for a single SQUID [22]. Using the definition of the average screening current (Eq. (3)), we can expand the model equations to second order in M,

$$\dot{\delta}_j^{(k)} = J + \frac{(-1)^j}{\beta_k}\left[\left(1 + \frac{2\pi}{\beta_k}\frac{\bar{M}}{N}\right)(\delta_1^{(k)} - \delta_2^{(k)} - 2\pi\Phi_0^{-1}\Phi_e) - \frac{2\pi}{\Phi_0}\bar{M}\bar{I}\right] - \sin\delta_j^{(k)} + \xi_k^{(j)} \tag{4}$$

where $\bar{M} = MN$. When $N \to \infty$ (thermodynamic limit), it is well known [23, 24] that models with mean-field coupling can be described by an evolution equation for the one-particle probability density, $\rho(\delta_1, \delta_2, t)$. Contrary to the case $N = 1$ where the FPE is linear, this probability density is the solution of the following *nonlinear* Fokker–Planck equation (NFPE):

$$\frac{\partial\rho}{\partial t} = D\left[\frac{\partial^2\rho}{\partial\delta_1^2} + \frac{\partial^2\rho}{\partial\delta_2^2}\right] - \frac{\partial}{\partial\delta_1}(v_1\,\rho) - \frac{\partial}{\partial\delta_2}(v_2\,\rho)\,, \tag{5}$$

subject to initial and boundary conditions (2π periodicity in δ_1 and δ_2). The drift-terms are given by

$$v_j(\delta_1, \delta_2, t) = J + \frac{(-1)^j}{\beta}\left(\delta_1 - \delta_2 - 2\pi n - 2\pi\Phi_0^{-1}\Phi_e - \frac{2\pi\bar{M}}{\Phi_0}\bar{I}\right) - \sin\delta_j\,, \quad j = 1, 2\,, \tag{6}$$

where n is an integer that ensures the 2π periodicity of the solution. The average screening current is given by

$$\bar{I}(t) = \int d\beta f(\beta) \int_0^{2\pi}\int_0^{2\pi} d\delta_1\, d\delta_2\, \frac{1}{\beta}(\delta_1 - \delta_2 - 2\pi n - 2\pi\Phi_0^{-1}\Phi_e)\rho(\delta_1, \delta_2, t) \tag{7}$$

where $f(\beta)$ represents a given probability distribution for β. For simplicity, in the numerical simulations that follow we have chosen squids with identical values of β: $f(\beta) = \delta(\beta - \beta_0)$.

The numerical simulation of the NFPE (5) was performed using a spectral method [21], which consists in a generalization for a nonlinear problem of the method presented in [22]. Upon the inclusion of the sinusoidal probe signal Φ_e, the average screening current $\bar{I}$ in the Fokker–Planck equation, after a transient, becomes purely sinusoidal containing harmonics (including the fundamental) of the probe signal frequency [21]. To investigate the effect of the probe signal on the system's response we have systematically varied the frequency. The result of this simulation is shown in Fig. 1(b), where we have plotted the amplitude of the oscillations in $\bar{I}$, $A_{\bar{I}}$, for two different values of $\bar{M}$. The amplitude shows a clear maximum as the probe frequency is varied, demonstrating the resonance effect of the probe signal. By comparing the two curves for different values of $\bar{M}$ in Fig. 1(b) we readily observe that a larger coupling $\bar{M}$ implies a slower running oscillation. Furthermore, the peak amplitude is larger as we approach the onset of the bifurcation (for this class of bifurcation the running frequency is identically zero at the critical point).

To demonstrate that the NFPE accurately captures the dynamics of a large system of globally coupled SQUIDs we have plotted the powerspectrum for $N = 500$ in the inset. Since the average screening current becomes sinusoidal, we can directly compare the peak in the powerspectrum, represented by a solid circle and a solid square, to $A_{\bar{I}}^2/2$. From this comparison, we can see clearly that the NFPE accurately describes the dynamics, at least for $N \geq 500$. For comparison, we note that one Langevin point in Fig. 1(b) required approximately 150 hours of CPU time while a single NFPE point required only 10 minutes of CPU time. Thus, the FPE approach represents a dramatic computational savings.

As a possible application we now propose to use this resonance phenomenon to detect weak "target" signals using coupled SQUID arrays. Recall that the frequency of the underlying running state varies as a function of the coupling strength. This means that as we vary the coupling parameter M the power output will increase as the frequency of the underlying running solution approaches the target frequency. Since the underlying running solution of a single SQUID has, most often, a higher frequency than the target signal, we investigated this scenario by applying a target signal with a frequency that is several orders smaller than the underlying frequency

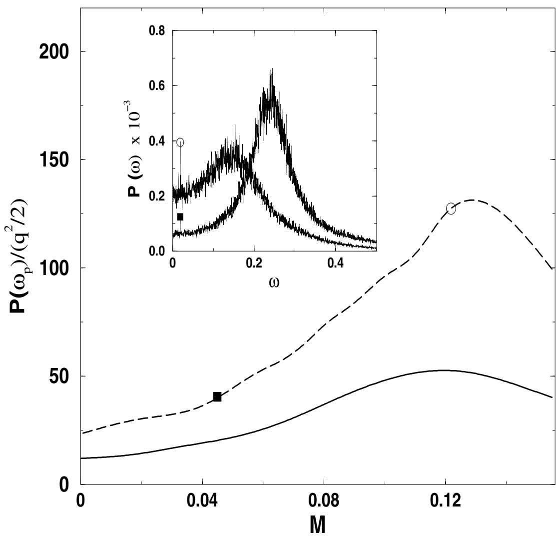

Fig. 2. Ratio between the peak in the powerspectrum and the signal strength as a function of the coupling strength M for a system of two coupled SQUIDs. The signal frequency is $\omega_p = 0.018$ with an amplitude of $q = 0.01$ (solid line) and $q = 0.0025$ (dashed line). The inset shows the powerspectrum for two different values of M ($M = 0.045$ for the closed square and $M = 0.122$ for the open circle). Other parameters as in Fig. 2.

of a single SQUID. Furthermore, keeping practical applications in mind, we considered only the case $N = 2$. In Fig. 2 we have plotted the amplification of the target signal, defined as the power at the target frequency (output) divided by the power of the target signal (input), as a function of the coupling strength. The curves, for two different values of q, clearly show a typical resonance shape, indicating the presence of an optimal value of M. The inset of Fig. 2 shows the powerspectrum for two different values of M. The peak values at ω_p are also plotted as symbols in the amplification curves. At the optimal of the coupling strength, the target signal for the small value of q is amplified by more than 100, representing a dramatic increase in sensitivity of the SQUID.

3. Conclusion

In conclusion, we have used a probe signal in a coupled system of noisy oscillators that allows us, via a resonance effect, to determine the underlying frequency of the system. For a system of coupled SQUIDs, we demonstrate how this effect, in conjunction with our finding that the coupling strength modifies the underlying frequency, can be utilized to create a sensitive detection device. It is important to stress that the probing resonance we have found here is not limited to the SQUID problem. We have added a probing signal to the noisy FitzHugh–Nagumo model exhibiting a supercritical Hopf bifurcation to a limit cycle and have found a similar sharpening of the peak in the powerspectrum [25] (see also [26, 27]). It would be interesting to investigate to what extent the probing resonance is a universal phenomenon within the wide class of coupled noisy oscillators.

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