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Uncertainty in phase-frequency synchronization of large populations of globally coupled nonlinear oscillators

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Abstract

Numerical evidence as well as an analytical description is given for the existence of a kind of "uncertainty principle" in the mean-field model (of the Kuramoto type) of large populations of nonlinearly coupled oscillators. Here the noise plays the role of the Planck constant. This means that there is a limit to the possibility of synchronizing both, phases and frequencies. The collective phase synchronization competes in fact, within such a model, with the frequency synchronization. The explanation rests on the effect of the noise, which is responsible for loss of accuracy. Comparison is made with a more sophisticated model, where such uncertainty is absent when the coupling parameter is sufficiently large. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Collective behavior, due to cooperative phenomena, leading to synchronization as opposite to incoherence, plays an important role in Physics, Biology [1,2], and Neural Networks [3]. These facts have been observed in diverse fields of science, such as superconducting Josephson junctions arrays [4–6], plasma waves [7], and phonon phenomena in condensed matter [8]. The reader is referred to [2,9,10] for other references and further applications. As is well known, a mathematical model which can capture a number of features due to cooperative phenomena is the following, consisting of a set on nonlinearly coupled differential equations,

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} K_{ij} f(\theta_j - \theta_i), \quad i = 1, 2, \dots, N, \tag{1}$$

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where $\theta_i(t)$ denotes the ith oscillator's phase, ω_i its natural frequency (picked up from a given distribution, $g(\omega)$), the K_{ij}'s; $K_{ij} > 0$, represent the coupling strengths, and $f(\theta)$ is a given periodic coupling function. In particular, Kuramoto [11,12] proposed a mean-field model with $K_{ij} = K/N$, $K > 0$, and a sinusoidal coupling function, which is analytically tractable:

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad i = 1, 2, \dots, N. \tag{2}$$

Other types of periodic coupling functions have been used, however, cf. [14–17]. System (2) is called a "mean-field" model in statistical mechanics, because each oscillator feels the influence of the entire population exactly as every other member. Indeed, it seems that this global coupling represents in a natural way the dynamical behavior of series arrays of Josephson junctions [4,5], laser arrays [18,19], and, to some extent, several biological phenomena, see [1], e.g., rather than being merely an approximation within the context of statistical mechanics. The research on nonlinear oscillators has been focused in recent years on the field of Josephson junctions arrays, in part because of their numerous important technological applications, e.g., high operation frequencies and voltage standards. Also, laser arrays are a subject of great current interest in applied physics, where a high optical power output is required. Recently, it has been found that the Josephson junctions *serial* arrays could be mapped into the Kuramoto model in Eq. (2) [4,5] (in the limit of zero capacitance, weak coupling, and weak disorder). On the other hand, it seems natural to introduce into such a model suitable, physically reasonable, noise terms, which take into account unavoidable imperfections and perturbations due to a variety of phenomena acting upon the system (e.g., thermal Johnson noise in the case of Josephson junctions). Indeed, Kuramoto himself [11,12] (see also [13]) proposed another model in which certain external random noises are present,

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) + \xi_i(t), \quad i = 1, 2, \dots, N, \tag{3}$$

where the ξ_i's are independent identically distributed white noises, with expected values

$$\langle \xi_j(t)\rangle = 0, \qquad \langle \xi_j(t)\xi_k(t')\rangle = 2D\delta(t-t')\delta_{jk}. \tag{4}$$

To measure the fraction of those oscillators synchronized in phase, the complex order-parameter

$$r\,\mathrm{e}^{\mathrm{i}\psi} = \frac{1}{N}\sum_{j=1}^{N}\mathrm{e}^{\mathrm{i}\theta_j} \tag{5}$$

has been introduced in Eq. (3). Eq. (3) can be then conveniently written as

$$\dot{\theta}_i = \omega_i + Kr\sin(\psi - \theta_i) + \xi_i(t), \quad i = 1, 2, \dots, N. \tag{6}$$

A neat picture of the case of very large N can be given by the limiting-model obtained when $N \to \infty$ (thermodynamic limit). In this case, the nonlinear partial integro-differential equation of the Fokker–Planck type,

$$\frac{\partial\rho}{\partial t} = D\frac{\partial^2\rho}{\partial\theta^2} - \frac{\partial}{\partial\theta}(v\rho), \tag{7}$$

where $D > 0$ measures the strength of the noise, and

$$v(\theta, \omega, t) = \omega + Kr\sin(\psi - \theta) \tag{8}$$

is the “drift velocity”, was derived (both formally [20], and later rigorously [21]). Here $\rho(\theta, \omega, t)$ is the one-oscillator probability density, and the order-parameter is now defined as

$$r\,\mathrm{e}^{\mathrm{i}\psi} = \int_0^{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\theta} \rho(\theta, \omega, t) g(\omega)\,\mathrm{d}\omega\,\mathrm{d}\theta, \tag{9}$$

where $g(\omega)$ is a given frequency distribution. Eq. (7) has to be considered along with suitable initial and boundary conditions (in particular, 2π-periodicity in θ), and normalization of the initial condition, $\int_0^{2\pi} \rho(\theta, \omega, t = 0)\,\mathrm{d}\theta = 1$, cf. [22].

While a meaningful synchronization phenomenon should concern both, *phase* and *frequency synchronization*, it seems that in the existing literature only phase synchronization has been rigorously analyzed. Indeed, synchronizing in frequency is of paramount importance, for instance, for the Josephson junctions arrays.

Strictly speaking, $\dot{\theta}_i$ should be called the “instantaneous frequency”, and v_i the “instantaneous drift”. However, so to say, as $\dot{\theta}_i$ has the same nature of the white noise, ξ_i, it can hardly be considered computable, in the present model. On the other hand, the drift v_i should be considered computable, once the solution θ_i has been evaluated; in fact, similarly to θ_i, v_i will be a continuous path stochastic process (like the Brownian motion). Moreover, by the same observation as above, it is only meaningful to evaluate macroscopic quantities related to $\dot{\theta}_i$, such as mean and variance. Now, it turns out that $E\dot{\theta}_i = Ev_i$, $\langle\dot{\theta}_i\rangle = \langle v_i\rangle$, where E denotes taking expected values and $\langle\cdot\rangle$ long-time averages. These equalities are due to the fact that the white noise has zero mean, and ensemble averages will coincide with time averages in the ergodic limit, which we assume to be valid. Besides, below (Section 2) we show that roughly $\Delta\dot{\theta}_i \geq \Delta v_i$, at least when $K \to \infty$. Therefore, we conclude that the instantaneous drift, v_i, could *effectively* replace the instantaneous frequency, $\dot{\theta}_i$, in assessing the *frequency* synchronization of the oscillator population.

In absence of noise, the fraction of the oscillators synchronized in frequency increases with the coupling strength, K. Note that $\dot{\theta}_i = v_i$, in this case. On the other hand, in the noisy case a discussion about frequency synchronization was implicitly made [11–13], observing that noise makes less clear the mutual entrainment, but such a phenomenon was not quantitatively assessed. In fact, in the presence of noise, it is possible to find the entire population perfectly synchronized in the long-term sense (according to $\langle\dot{\theta}_i\rangle$), while the instantaneous drift synchronization (according to v_i) gets worse when the coupling strength increases.

In this paper, we consider the possibility of a simultaneous synchronization in phase *and* frequency (instantaneous drift *and* long-term average), in the finite-size system (N finite), governed by Eq. (3), as well as in the case of infinitely many oscillators, governed by Eq. (7). To this purpose, we computed the spread of both phase and frequency, as a function of the noise and the coupling strength.

A major result, whose evidence can be supported analytically and numerically, is that a kind of “uncertainty principle” (similar to the Heisenberg principle in Quantum Mechanics, concerning pairs of conjugate variables), involving here the spread of phase as well as instantaneous drift, can be observed. The uncertainty principle, derived in the framework of Stochastic Mechanics, can be found, e.g., in [23]. Indeed, when a state of greater phase synchronization is obtained, the dynamical behavior of the system leads it to a state of lower degree of frequency synchronization (intended in the latter sense), and conversely. In other words, if we wish to synchronize the oscillators in phase increasing the coupling strength, the corresponding effect is to desynchronize in the instantaneous drift. Analytical as well as numerical simulations supporting such conclusions are given in Sections 2 and 3. Numerical simulations also show that the spread of the long-term average frequency distribution decreases, which corresponds to long-term frequency synchronization. Confining only to the long time average frequency synchronization, in fact, one should obtain a measure of the “error” (the spread), given by its variance.

Clearly, in the presence of noise, a macroscopic observable is rather given by an average over a large number of realizations of noise. In particular, the measure of the frequency of any single oscillator is given by the average above, or equivalently over long times (in the ergodic limit), and it is also necessary to consider *simultaneously* the error made when the average over a long time is computed, which is given by the variance of the instantaneous frequency distribution. More precisely, in Section 2 we compute analytically and numerically $\Delta\dot{\theta}_i$ in the finite size case, and Δv, in the limit of infinitely many oscillators, when $K \to \infty$, in which case we expect to find the best synchronization in phase. At the end of Section 2 a model free of uncertainty is presented, i.e., where the error in the measure of the frequency is bounded and indeed independent of the coupling strength. In Section 3, numerical results supporting the evidence of the aforementioned behavior are given, and in Section 4 the high points of the paper are summarized.

In closing, it should be emphasized that the "uncertainty principle" described here does *not* represent a peculiarity of populations of globally coupled oscillators. A similar phenomenon can be observed even for a *single* noisy oscillator subject to a periodic external force, depending on θ, or for a particle in a one-dimensional potential well, affected by noise. Such an uncertainty, however, plays an important role in globally coupled oscillator populations, with respect to the *synchronization* phenomenon. For a single particle, uncertainty means that measuring its angular position with a given accuracy prevents measuring its frequency with an arbitrary precision. In a many-particle system the meaning of uncertainty is different, in that the larger is the number of particles having a very close angular positions, the smaller is the number of them with close frequencies.

2. Analytical results

In the following, we analyze the statistical behavior of the drift velocity, v (see Eq. (8)), which, in the limit of infinitely many oscillators, represents a measure of the instantaneous frequency. Introducing a frequency distribution, $\sigma(v, \omega, t)$, along with the phase distribution, $\rho(\theta, \omega, t)$ (the solution to Eq. (7)), we find

$$\sigma(v, \omega, t)\,\mathrm{d}v = \rho(\theta(v), \omega, t)\left|\frac{\mathrm{d}\theta}{\mathrm{d}v}\right|\mathrm{d}v = \rho\left(\psi - \sin^{-1}\left(\frac{v-\omega}{Kr}\right), \omega, t\right)\frac{1}{((Kr)^2-(v-\omega)^2)^{1/2}}\,\mathrm{d}v. \tag{10}$$

It is also convenient to introduce, along with the order-parameter defined in Eq. (9) (which refers to the phase synchronization), a similar quantity, to describe the analogous behavior for the frequency synchronization. Such an order-parameter is defined by

$$s\,\mathrm{e}^{\mathrm{i}\phi} = \int_{-\infty}^{+\infty}\int_{\omega-Kr}^{\omega+Kr} \mathrm{e}^{\mathrm{i}v}\sigma(v, t, \omega)g(\omega)\,\mathrm{d}\omega\,\mathrm{d}v. \tag{11}$$

In addition to the numerical simulations described below, it is possible to give an analytical evidence of such a complementary phase and frequency dynamical behavior. In the following, for simplicity, we suppose the oscillators to be *identical*, $g(\omega) = \delta(\omega)$; this will not be much of a restriction, in view of the large values of the coupling strength, K, we are interested in. In fact, when $K \to \infty$, the effect of the specific frequency distribution, $g(\omega)$, will not be apparent, cf. [24]. The *stationary* solution to Eq. (7), say $\rho(\theta)$, is given, in this case, by

$$\rho(\theta) = \frac{\exp[(Kr/D)\cos(\psi-\theta)]\int_0^{2\pi}\exp[(-Kr/D)\cos(\psi-\theta-\beta)]\,\mathrm{d}\beta}{\int_0^{2\pi}\int_0^{2\pi}\exp[(Kr/D)\cos(\psi-\theta)]\exp[(-Kr/D)\cos(\psi-\theta-\beta)]\,\mathrm{d}\beta\,\mathrm{d}\theta}. \tag{12}$$

Making use of the Laplace method, it is possible to find an asymptotic solution to Eq. (12), in the limit of large couplings, $K \to \infty$, which is

$$\rho(\theta) = \sqrt{\frac{Kr}{2\pi D}}\exp\left(-\frac{Kr}{D}(1-\cos(\psi-\theta))\right)\left(\frac{1}{1+(D/8Kr)} + \mathrm{O}\left(\frac{1}{K^2}\right)\right). \tag{13}$$

The limiting-case $K \to \infty$ is important in that strong phase synchronization is expected under such a condition, cf. [9,10,25]. Knowing the phase distribution, the instantaneous drift distribution (which is also the drift distribution of the entire oscillator population at each time) can be evaluated as

$$\sigma(v) = \sqrt{\frac{Kr}{2\pi D}} \frac{1}{((Kr)^2 - v^2)^{1/2}} \exp\left(-\frac{1}{D}(Kr - ((Kr)^2 - v^2)^{1/2})\right)\left(\frac{1}{1 + (D/8Kr)} + \mathrm{O}\left(\frac{1}{K^2}\right)\right), \tag{14}$$

cf. Eqs. (10) and (13), setting $\omega = 0$ (because $g(\omega) = \delta(\omega)$). Consider the spread of both, phase and drift distributions, defined by

$$(\Delta\theta)^2 = \langle(\theta - \psi)^2\rangle_\rho - (\langle\theta - \psi\rangle_\rho)^2, \qquad (\Delta v)^2 = \langle v^2\rangle_\sigma - (\langle v\rangle_\sigma)^2, \tag{15}$$

respectively, where $\langle\cdot\rangle_{\rho,\sigma}$ denotes taking averages over the phase or the drift distribution. Taking into account the symmetry properties of $\rho(\theta)$ and of $\sigma(v)$, we get $\langle\theta - \psi\rangle_\rho = \langle v\rangle_\sigma = 0$, and using again the Laplace method to evaluate the averages when $K \to \infty$, Eq. (15) yields

$$(\Delta\theta)^2 = \frac{D}{K}\frac{1}{1 + (D/8Kr)} + \mathrm{O}\left(\frac{1}{K^2}\right), \qquad (\Delta v)^2 = \frac{KD}{1 + (D/8Kr)} + \mathrm{O}\left(\frac{1}{K^2}\right). \tag{16}$$

Note that, when K becomes larger and larger, the phase spread decreases, while the drift spread increases. Multiplying these two relations side by side leads to

$$\Delta\theta\Delta v \sim D, \qquad K \to \infty, \tag{17}$$

which gives evidence of the aforementioned "uncertainty principle", and shows that noise plays a crucial role within the synchronization phenomena.

In the case of an arbitrary value of the coupling strength, K, it is possible to derive a general expression for the drift spread, which depends on the order parameter, $r > 0$. From the definition of average over the drift distribution and Eq. (10), one can evaluate $\langle v\rangle_\sigma$,

$$\langle v\rangle_\sigma = \int_{-Kr}^{Kr} \mathrm{d}v\, v\sigma(v) = \int_0^{2\pi} \mathrm{d}\theta\, Kr \sin(\psi - \theta)\rho(\theta), \tag{18}$$

but the last integral vanishes by the definition of the order parameter, cf. Eq. (9), thus $\langle v\rangle_\sigma = 0$. Similarly, $\langle v^2\rangle_\sigma$ is given by

$$\langle v^2\rangle_\sigma = \int_{-Kr}^{Kr} \mathrm{d}v\, v^2\sigma(v) = \frac{K^2r^2}{2}\int_0^{2\pi} \mathrm{d}\theta\,[1 - \cos(2(\psi - \theta))]\rho(\theta). \tag{19}$$

The last integral corresponds to the second moment of the probability density, $\rho(\theta)$, which, in case of stationary solutions, can be easily evaluated from the hierarchy of Eqs. (51)–(53) in [24]. The result is given by

$$\int_0^{2\pi} \cos(2(\psi - \theta))\rho(\theta)\,\mathrm{d}\theta = 1 - \frac{2D}{K}. \tag{20}$$

Therefore, the spread in the drift distribution is

$$\langle v^2\rangle_\sigma \equiv (\Delta v)^2 = KDr^2, \tag{21}$$

which increases monotonically with K. When K is close to $K_\mathrm{c} = 2D$, with $K \geq K_\mathrm{c}$, because we are interested in solutions (partially) synchronized in phase, the order parameter r can be evaluated perturbatively [25], and is given by

$$r = \left(\frac{K - K_\mathrm{c}}{D}\right)^{1/2} + \mathrm{O}(|K - K_\mathrm{c}|). \tag{22}$$

Introducing (22) into (21), we obtain

$$(\Delta v)^2 \approx K(K-2D), \qquad K \approx K_c = 2D. \tag{23}$$

To evaluate the spread in phase when $K \approx K_c$, it is necessary to analyze $\rho(\theta)$ in such a limit. To this purpose, we expand ρ in Fourier series, truncating the series to the second harmonic,

$$\rho(\theta) \approx \frac{1}{2\pi} + \sum_{n=1}^{2}[c_n \cos(n(\psi-\theta)) + s_n \sin(n(\psi-\theta))]. \tag{24}$$

This approximation seems reasonable in view of the fact that $K \approx K_c$ (cf. [26]). Inserting (24) in (7), we obtain

$$\rho(\theta) \approx \frac{1}{2\pi} + \frac{2((K-K_c)D)^{1/2}}{\pi K}\cos(\psi-\theta) + \left(1-\frac{2D}{K}\right)\frac{1}{\pi}\cos(2(\psi-\theta)). \tag{25}$$

Then, the spread in phase is given by

$$(\Delta\theta)^2 \approx \frac{\pi^2}{3} - \frac{8((K-K_c)D)^{1/2}}{K} + \left(1-\frac{2D}{K}\right), \qquad K \approx K_c = 2D. \tag{26}$$

The main conclusion is that the drift spread always increases when K grows, while the spread in phase decreases correspondingly.

In addition, a similar analysis can be carried out for the case of a finite-size system, governed by Eq. (3), in the limit of infinitely many oscillators. To this purpose, we compute the instantaneous frequency variance through *time averages*, in the ergodic limit when time goes to infinity. Note that such a variance sizes the error made in measuring the time averages of the instantaneous frequency. In practice, one can only measure $\langle\dot\theta_i\rangle \pm \Delta\dot\theta_i$. Thus,

$$(\Delta\dot\theta_i)^2 = \langle(\dot\theta_i)^2\rangle - \langle\dot\theta_i\rangle^2, \tag{27}$$

where

$$\langle(\dot\theta_i)^2\rangle = \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau (\dot\theta_i)^2\,\mathrm{d}t, \tag{28}$$

$$u \equiv \langle\dot\theta_i\rangle = \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau \dot\theta_i\,\mathrm{d}t, \tag{29}$$

the initial time $t=0$ being chosen after transients have decayed. In case of identical oscillators, Eq. (29) can be easily evaluated, in the ergodic limit and when $N\to\infty$,

$$\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau \dot\theta_i\,\mathrm{d}t = Kr\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau \sin(\psi-\theta_i)\,\mathrm{d}t = Kr\int_0^{2\pi}\sin(\psi-\theta)\rho(\theta)\,\mathrm{d}\theta. \tag{30}$$

Recall that $\omega_i \equiv 0$, being $g(\omega)=\delta(\omega)$, and that $r(t)\sim$ constant >0 as $t\to+\infty$, because of the assumed stationarity. In fact, it has been shown [9,10] that when the frequency distribution is unimodal, $r(t)$ can be either identically zero (in the incoherent case), or tend to a positive constant, when $t\to+\infty$. From the definition of the order-parameter in Eq. (9), taking the imaginary part, we find that this integral is identically zero. On the other hand, Eq. (28) becomes

$$\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau(\dot\theta_i)^2\,\mathrm{d}t = \lim_{\tau\to\infty}\left[\frac{K^2r^2}{\tau}\int_0^\tau \sin^2(\psi-\theta_i)\,\mathrm{d}t + \frac{1}{\tau}\int_0^\tau \xi_i^2(t)\,\mathrm{d}t + \frac{2Kr}{\tau}\int_0^\tau \xi_i(t)\sin(\psi-\theta_i)\,\mathrm{d}t\right]. \tag{31}$$

The last integral on the right-hand side can be evaluated using Ito's formula [27]. The result is given by

$$\frac{1}{\tau}\int_0^\tau \xi_i(t)\sin(\psi-\theta_i)\,\mathrm{d}t = \frac{1}{\sqrt{2D}}\frac{\cos(\psi-\theta_i(\tau)) - \cos(\psi-\theta_i(0))}{\tau} - \frac{1}{\sqrt{2D}}Kr\frac{1}{\tau}\int_0^\tau \sin^2(\psi-\theta_i)\,\mathrm{d}t + \frac{D}{\sqrt{2D}}\frac{1}{\tau}\int_0^\tau \cos(\psi-\theta_i)\,\mathrm{d}t. \quad (32)$$

Again, using ergodicity and $N \to \infty$, the integrals on the right-hand side become

$$\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau \cos(\psi-\theta_i)\,\mathrm{d}t = \int_0^{2\pi}\cos(\psi-\theta)\rho(\theta)\,\mathrm{d}\theta = r, \quad (33)$$

$$\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau \sin^2(\psi-\theta_i)\,\mathrm{d}t = \int_0^{2\pi}\sin^2(\psi-\theta)\rho(\theta)\,\mathrm{d}\theta = \frac{1}{2} - \frac{1}{2}\int_0^{2\pi}\cos(2(\psi-\theta))\rho(\theta)\,\mathrm{d}\theta. \quad (34)$$

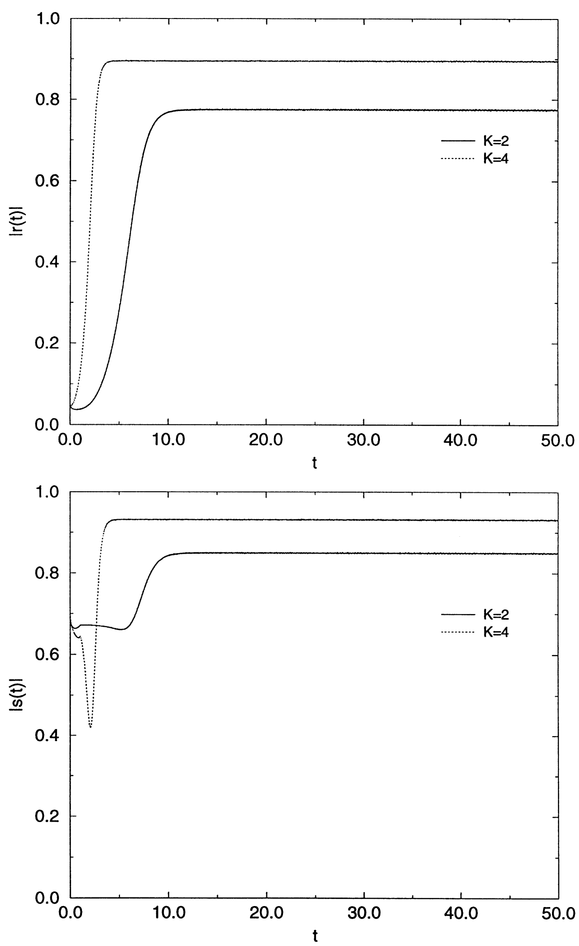

Fig. 1. Finite-size model ($N = 20\,000$ oscillators) without noise ($D = 0$): time evolution of the order-parameter amplitudes $|r(t)|$ and $|s(t)|$. The frequency distribution is the Lorentzian $g(\omega) = (\gamma/\pi)/(\gamma^2+\omega^2)$ with $\gamma = 0.4$, and the coupling strength is $K = 2$ (solid line), or $K = 4$ (dotted line).

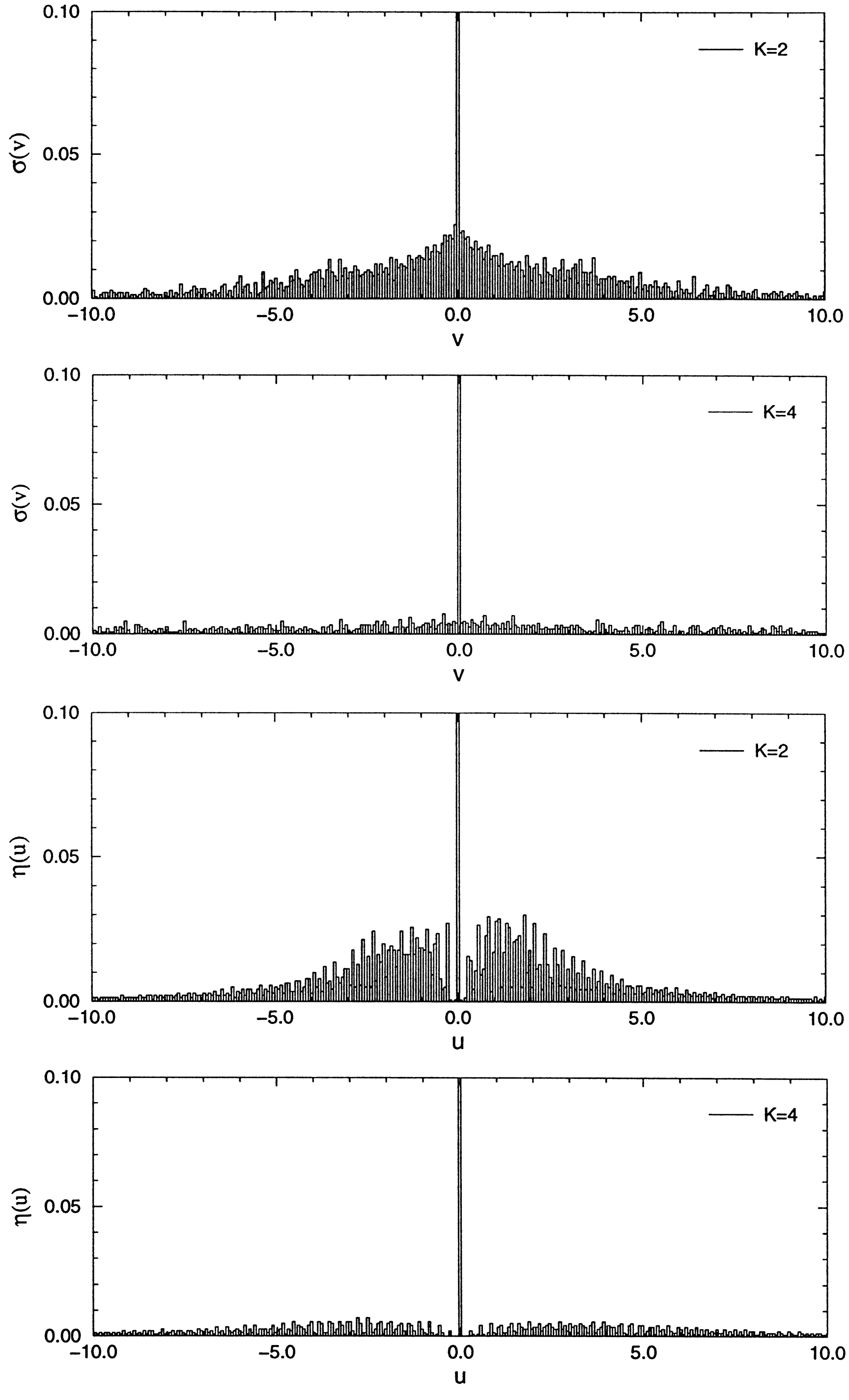

Fig. 2. Finite-size model without noise ($D = 0$): drift velocity ($\sigma(v)$) and long-time averaged frequency ($\eta(u)$) distributions. Parameters are the same as in Fig. 1. Note that synchronization in phase and frequency increases with the coupling strength K.

Finally, introducing all these expressions into Eq. (31), we obtain

$$\langle(\dot{\theta}_i)^2\rangle = Kr^2 D + \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau \xi_i^2(t)\,\mathrm{d}t \geq Kr^2 D. \tag{35}$$

Compare this result, i.e. $(\Delta\dot{\theta}_i)^2 = \langle(\dot{\theta}_i)^2\rangle \geq Kr^2 D \sim KD$, when $K \to \infty$ (since $r \sim 1$ in this limit), with $(\Delta v)^2 \sim KD$ when $K \to \infty$ in Eq. (16). We stress that the long-time averaged instantaneous frequency, u in Eq. (29), has been found to be zero. The associated error, $\Delta\dot{\theta}_i$, on the other hand, grows unboundedly as $K \to \infty$, which fact makes it doubtful whether the quantity $\langle\dot{\theta}_i\rangle$ itself ($\langle\dot{\theta}_i\rangle = \langle v_i\rangle$) might be precisely measured (and thus the frequency synchronization realized). Clearly, the whole procedure above, involving squares of distributional stochastic processes such as the white noises $\xi_i(t)$'s, and the $\dot{\theta}_i(t)$'s, is formal.

In order to overcome the uncertainty phenomenon affecting the Kuramoto model in Eq. (3), a more sophisticated description has been proposed in [28]. The finite size system without noise in Eq. (2) has been generalized by Ermentrout [29] and Tanaka et al. [30,31], to take into account, for instance, a more realistic (slower) approach to the synchronized state, in certain populations of fireflies. This improvement includes inertial effects, thus leading to a system of *second-order* equations (rather than first-order equations as in Eq. (3)), and it seems to better describe also the dynamical behavior of superconducting Josephson junctions arrays at nonzero temperature and with nonzero capacitance. Here is the system:

$$m\ddot{\theta}_i + \dot{\theta}_i = \Omega_i + Kr_N \sin(\psi_N - \theta_i) + \xi_i(t), \quad i = 1, \dots, N, \tag{36}$$

where now Ω_i is the natural frequency of the ith oscillator (instead of ω_i in the Kuramoto model, Eq. (3)), and $m > 0$ represents the aforementioned inertial term.

In superconducting Josephson arrays, the so-called McCumber parameter, which is a dimensionless constant related to the internal capacitance of the Josephson junction, plays the role of an inertial term. The governing equations are given by

$$\beta\ddot{\phi}_j + \dot{\phi}_j + \sin\phi_j = i_j + \xi_j, \quad j = 1, 2, \dots, N, \tag{37}$$

β being the McCumber parameter, ϕ_j the phase difference of the jth junction, i_j the critical current, and ξ_j the noise term introduced to take into account thermal fluctuations. Depending on the type of junction (i.e., size, geometry,

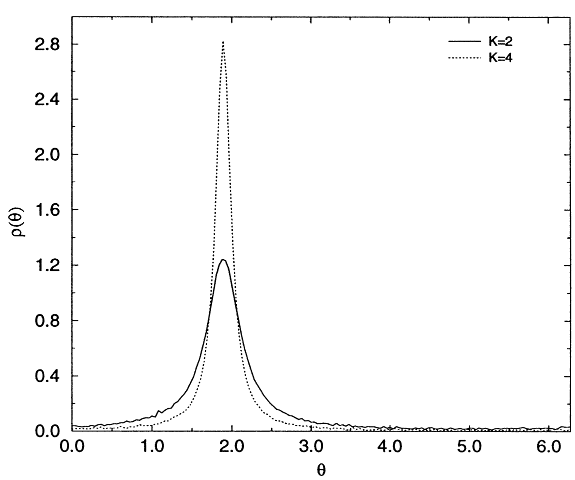

Fig. 3. Finite-size model without noise ($D = 0$): phase distribution. Parameters are the same as in Fig. 1.

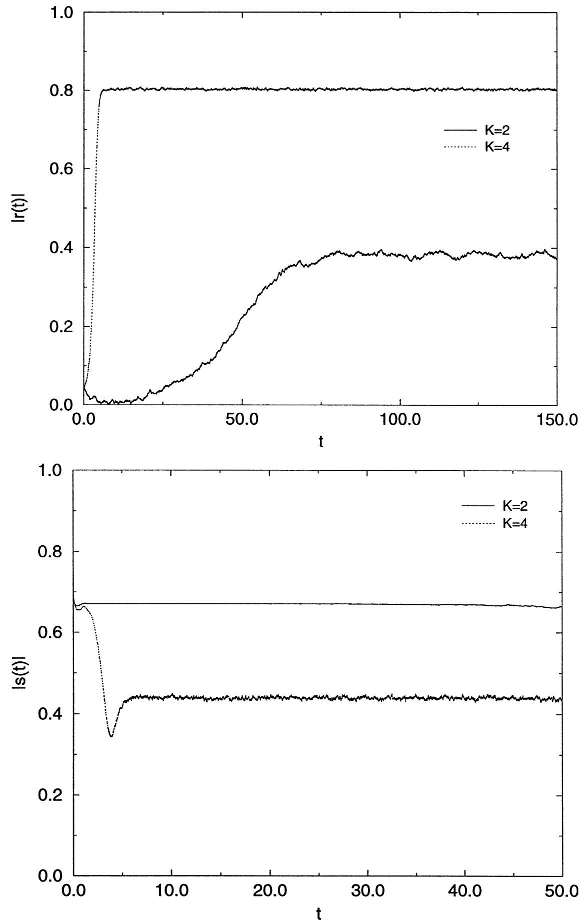

Fig. 4. Finite-size model ($N = 20\,000$ oscillators) with noise ($D = 0.5$): time evolution of the order parameter amplitudes $|r(t)|$ and $|s(t)|$. The frequency distribution is the Lorentzian $g(\omega) = (\gamma/\pi)/(\gamma^2 + \omega^2)$ with $\gamma = 0.4$. Parameters are now $K = 2$ (solid line), and $K = 4$ (dotted line).

and type of coupling), the value of β can range approximately from 10^{-6} to 10^7. Therefore, there are indeed cases where the value of β cannot be neglected, and is necessary to include the inertial terms in the model equations.

Starting from such a system, in [28] a nonlinear partial parabolic differential equation has been obtained in the limit of infinitely many oscillators, $N \to \infty$. Such an equation, i.e.,

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial \omega^2} - \frac{1}{m} \frac{\partial}{\partial \omega} [(-\omega + \Omega + Kr \sin(\psi - \theta))\rho] - \omega \frac{\partial \rho}{\partial \theta}, \tag{38}$$

generalizes the Kuramoto equation (cf. Eq. (7)). Note that here $\rho(\theta, \omega, \Omega, t)$ denotes the one-oscillator probability density, Ω is the natural frequency picked up from a given distribution, and the independent variable ω corresponds to the instantaneous frequency. The order parameter, however, becomes now

$$r\, \mathrm{e}^{\mathrm{i}\psi} = \int_{-\infty}^{+\infty} \mathrm{d}\omega \int_0^{2\pi} \mathrm{d}\theta \int_{-\infty}^{+\infty} \mathrm{d}\Omega\, g(\Omega)\, \mathrm{e}^{\mathrm{i}\theta} \rho(\theta, \omega, \Omega, t). \tag{39}$$

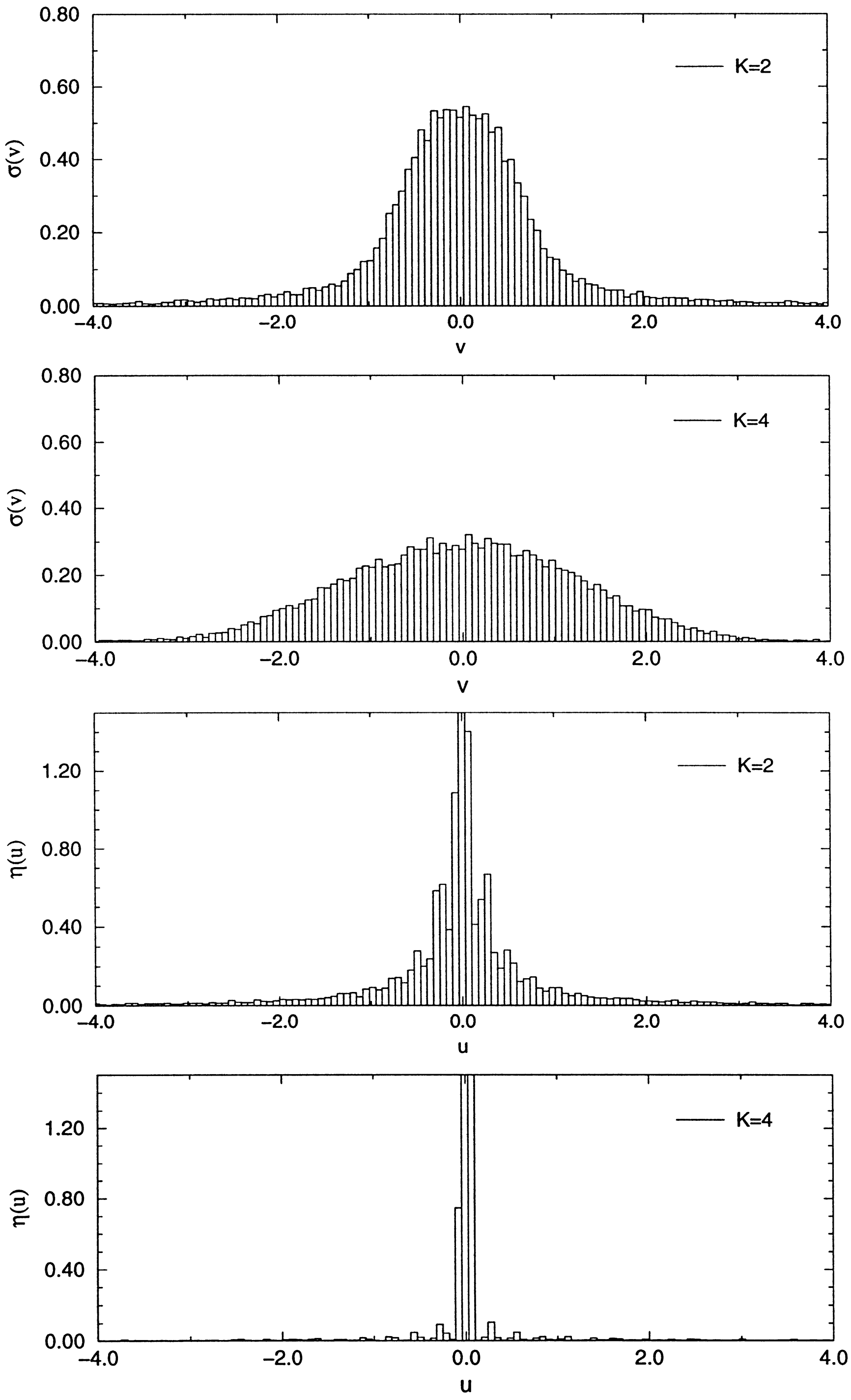

Fig. 5. Finite-size model with noise ($D = 0.5$): drift velocity ($\sigma(v)$) and long-time averaged frequency ($\eta(u)$) distributions. Parameters are the same as in Fig. 3.

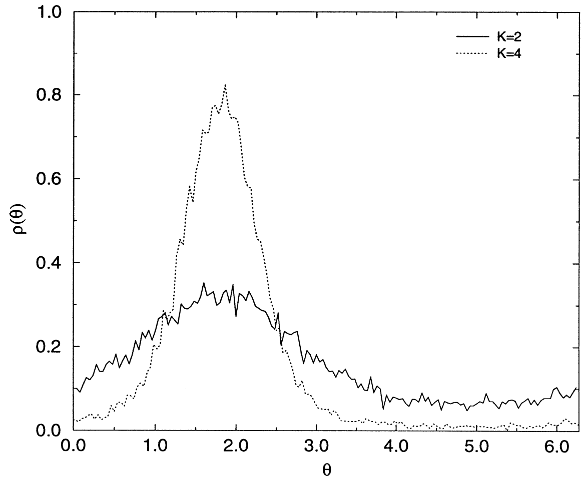

Fig. 6. Finite-size model with noise ($D = 0.5$): phase distribution. Parameters are the same as in Fig. 3.

For the model in Eq. (38), the relations

$$(\Delta\theta)^2 \sim \frac{D}{K}, \qquad (\Delta\omega)^2 \sim \frac{D}{m}, \qquad \Delta\theta\Delta\omega \sim \frac{m^{-1/2}D}{K^{1/2}}, \quad K \to \infty, \tag{40}$$

have been found [28], where the underlying averages were *all* taken with respect to ρ, instead of Eqs. (16) and (17). It is clear that now there will be *no uncertainty* when $K \to \infty$, and that the error in measuring ω is bounded in such a limit (indeed, it is indepedent of K). We stress that, in the Kuramoto model, phase and frequency are dependent on each other, while in the "Ermentrout–Tanaka et al." model they are independent. In the Kuramoto model, Eq. (3), evaluating $\langle\dot{\theta}_i\rangle$ amounts to evaluating $\langle v_i\rangle$, which, in turn, requires obtaining θ_i. This shows that (time-averaged) frequency and phase depend on each other. In the "Ermentrout–Tanaka et al." model, instead, *both*, θ_i and $\dot{\theta}_i$, can be evaluated, and this *independently* of each other. It is this fact that explains why synchronizing in both, phase and frequency, is possible in the latter case.

In the singular limit for $m \to 0$, the system in Eq. (36) reduces to the Kuramoto's model, Eq. (3), as it can be shown by Nelson's Theorem 10.1 in [32]. Similarly, in the infinite population case, one can recover Kuramoto's model (Eq. (7)) from Eq. (38), invoking Nelson's "pseudotheorem" 10.2 in [32].

3. Numerical results

We have simulated numerically both, the Langevin system in Eq. (3) and the noiseless system obtained from it dropping the noise terms, i.e., Eq. (2). To this purpose, we conducted numerical simulations of the Monte Carlo-type of a large number of Langevin equations (N=20 000 oscillators). From Figs. 1–3, the case without noise is considered, while from Figs. 4–8, the noisy case is analyzed. In Fig. 1, the time evolution of both order parameter amplitudes, $r(t)$ and $s(t)$, are plotted, showing that increasing the coupling strength, $K = 2$ (solid line) to $K = 4$ (dotted line) improves the degree of synchronization *in both*, phase and frequency. In Fig. 2, the distribution of both, drift velocity ($\sigma(v)$) and long-time average frequency ($\eta(u)$, u being defined in Eq. (29)), have been computed, exhibiting the same behavior as in Fig. 1. The phase distribution is shown in Fig. 3. A Lorentzian frequency distribution has been used through Figs. 1–6, while in Figs. 7 and 8 the unimodal $g(\omega) = \delta(\omega)$ was adopted.

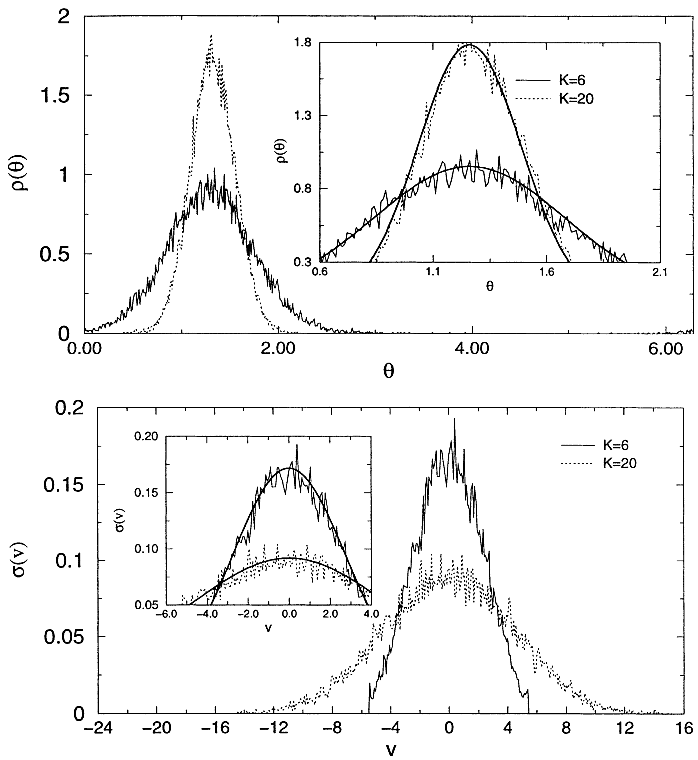

Fig. 7. Phase and frequency distributions. Comparison between the analytical solution (thick solid line in the inset), obtained by the Fokker–Planck equation (cf. Eqs. (13) and (14)), and the numerical solution, for two different coupling strengths, $K = 6$ (solid line), and $K = 20$ (dotted line), $D = 1$, and $g(\omega) = \delta(\omega)$.

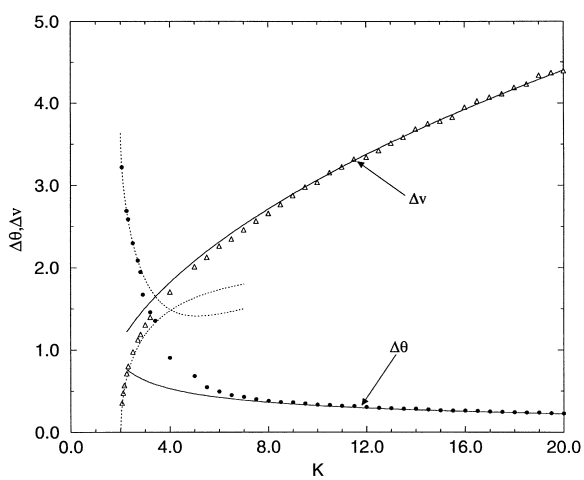

Fig. 8. Comparison between the analytical and numerical spread in phase and frequency, as functions of the nonlinearity parameter K. The analytical spread has been obtained asymptotically when $K \to \infty$ (solid line), and perturbatively when $K \approx K_c = 2D$ (dotted line). Numerical spread: (△) Δv, (●) $\Delta\theta$. $D = 1$ is kept fixed.

When some amount of noise enters the system, changes in the dynamical behavior are dramatic. In Fig. 4, increasing K improves the phase synchronization, measured by $r(t)$, while the drift frequency synchronization, measured by $s(t)$ gets worse. The fluctuations affecting the behavior of $|r(t)|$ for $K = 2$, are due to the fact that such a value of K is rather close to the critical value, K_c ($K_c = 1.8$ in this case). In Fig. 5, the distribution of both types of frequencies, v and u, is plotted. Note that the spread of the instantaneous drift velocity distribution increases when the coupling strength increases, while the spread of the long-term frequency distribution decreases. We stress that frequency synchronization in the sense of the $\eta(u)$ distribution overlooks the error (given by the spread in the instantaneous drift distribution) made when merely the long time average is considered.

The *stationary* phase distribution is shown in Fig. 6. In Fig. 7, a comparison between the analytical and numerical distributions in both phase and frequency is shown, for two different values of the coupling strength. In Fig. 8, the analytical and numerical spreads in phase and frequency are shown, as function of the nonlinearity parameter, K, when the noise is kept fixed. Such behavior yields further evidence of the uncertainty phenomenon as introduced in the paper.

4. Conclusions

In closing, we summarize the high points of the paper. An “uncertainty principle” has been found to govern phase and frequency synchronization, where phase and frequency play the role of conjugate variables in Quantum Mechanics. We stress that such an uncertainty is by no means peculiar of globally coupled oscillator systems, but it is important in this context because it strongly affects the possibility of synchronizing (in the framework of the model analyzed here). Numerical simulations as well as asymptotic relations support such conclusions. Uncertainty problems have not been encountered in a more elaborated model of interest to both populations of fireflies and superconducting Josephson junctions arrays. In the latter case, recall that “instantaneous frequency”, $\dot{\theta}_i$, has the meaning of voltage, say V_i, so that $\langle V_i \rangle$ represents the DC component of it. Therefore, focusing the attention on such time averages, one would neglect all information about the fluctuating components of the voltage, which size the error affecting the DC component. Such an error, being unbounded, in the Kuramoto model, as the coupling strength grows unboundedly, could make meaningless the measure of $\langle V_i \rangle$ itself. While, in practical experiments, V_i itself can be measured [33], in order to compare experimental results to the Kuramoto model one should actually measure $\langle V_i \rangle$. This is equivalent to measuring the mean drift $\langle v_i \rangle$, and thus (by its definition), measuring θ_i. Such an *indirect* method, however, implies a large amplification of the error made in the measure of θ_i.

All the previous discussions could be referred to *a single* Josephson junction (a single oscillator). Thinking of a large array, consisting of many junctions, however, the uncertainty implies that the larger is the number of oscillators with very close phases, the smaller is the number of them with close instantaneous drifts.

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