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Spectral analysis and computation for the Kuramoto–Sakaguchi integroparabolic equation

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A spectral method is developed to numerically solve the so-called Kuramoto–Sakaguchi equation, which is a nonlinear integro-differential equation of the parabolic type, governing the dynamical statistical behaviour of certain populations of nonlinearly coupled random oscillators. The approach rests on explicit bounds for the space derivatives of solutions, obtained via energy-like estimates. Bounds for the numerical approximations of solutions are given, and improved (sometimes appreciably) by means of an '*a posteriori* error analysis'. Plots are shown to illustrate the performance of the method, and comparison with a finite difference approach is also made.

Keywords: spectral method; nonlinear parabolic equations; integro-differential parabolic equations; populations of coupled oscillators; Kuramoto–Sakaguchi equation.

Introduction

Compared to the body of literature devoted to the numerical treatment of partial (purely) differential equations of the parabolic type, the scientific production concerning similar equations but including *integral terms*, is much more limited. Within such a class, some papers are concerned with integrals *with respect to time*, thus describing, for instance, memory effects in materials (see, for example, Larsson *et al.* (1998) and references therein), while others treat the equally interesting case of integral terms *with respect to space* variables, which may take into account global effects, such as distributed phenomena acting on the system (see, for example, Pao (1992), for some general models, and Fairweather & Saylor (1991) and references therein, Fairweather & López-Marcos (1991), Fairweather & López-Marcos (1994), Sun (1996), Spigler & Vianello (1994), Spigler & Vianello (1995) for the relevant numerical treatment). A number of phenomena arising in such diverse areas as vibration and heat conduction, nuclear reactor dynamics, chemical diffusion, and thermoelasticity, can indeed be modelled by partial integrodifferential equations of parabolic or hyperbolic type, cf. Fairweather & Saylor (1991). Besides, recall the important case of the Boltzmann equation and of other transport equations, where integration with respect to velocity (often much similar to a space variable) appears.

Numerous phenomena pertaining to physics, biology, medicine and neural networks, not mentioned in Fairweather & Saylor (1991), are reasonably described in terms of large populations of nonlinearly coupled, often noisy, oscillators. From a mathematical standpoint, this amounts to facing high-dimensional systems of nonlinear, possibly stochastic, differential equations. Among the aforementioned phenomena, one can find populations of insects and pacemaker heart cells, neurons, and Josephson junction arrays, just to give some examples, cf., for example, Strogatz & Mirollo (1991). When the interaction among the members of the populations above is of the so-called 'mean field' type, that is when each member of the population feels a global effect due to the others in the same way, then the system of equations modelling their dynamical behavior can be reduced, in the limit of infinitely many oscillators, to a single nonlinear partial differential equation, called Kuramoto (or Kuramoto–Sakaguchi) equation (Kuramoto 1975; Sakaguchi 1988; Strogatz & Mirollo 1991). In such an equation, which governs the time evolution of the one-oscillator probability density, the global, mean field interaction appears through an integral term. This integral is in the space variable. In the Kuramoto–Sakaguchi equation there is an additional variable, the natural frequency of the oscillators, with respect to which no derivatives appear, but the integral is also made with respect to it. It seems that the Kuramoto–Sakaguchi equation does *not* fall within the classes considered in the previous literature.

In Lavrentiev & Spigler (2000), the Kuramoto–Sakaguchi equation has been analyzed, and existence and uniqueness of classical solutions have been established. Also, some rather formal computation of its solutions have been obtained in Sartoretto *et al.* (1998), and here and there in the physical literature (see, for example, Acebrón *et al.* 1998; Bonilla *et al.* 1992, 1998). In this paper we derive estimates for the space derivatives of the solution to the Kuramoto–Sakaguchi equation, thus allowing for its spectral numerical treatment, in a rigorous way. Spectral methods, based on the representation of solutions as a series of given, suitably smooth, basic functions, have been proved to be very efficient, since the error typically decreases exponentially when increasing the number of the basic functions used, the number of 'harmonics' (when approximating analytic solutions), and in view of the possibility of rapid summation often available, cf., for example, Fornberg & Sloan (1994); Fornberg (1996); Gottlieb & Orszag (1977). In the special case of *periodic* solutions (that is the case of interest to the Kuramoto–Sakaguchi equation), the natural candidates for the basic functions are trigonometric functions, and in this case the celebrated Fast Fourier Transform (FFT) can be used for the latter purpose. General ideas as well as even recent applications can be found, for example, in Boyd (1989), Canuto *et al.* (1988), Fornberg & Sloan (1994); Fornberg (1996), Funaro (1992), Gottlieb & Orszag (1977). In this paper we describe a spectral method for the numerical solution of the Kuramoto–Sakaguchi equation. Pointwise as well as global L^2 error estimates are obtained, and some advantage of an *a posteriori* analysis is shown, both theoretically and numerically.

The paper is arranged as follows. In Section 1, we recall the problem and show the form of the error obtained when adopting a spectral algorithm; in Sections 2 and 3, we obtain few estimates for the space derivatives, needed for estimating in turn the error term above; recursions for obtaining estimates for such derivatives at all orders are then worked out in Sections 4 and 5. In Section 6, the important special case of the stationary solutions is considered and the relevant estimates are obtained. In Section 7, several pictures illustrate the numerical results, and a comparison with those produced by finite difference discretizations are shown. Throughout, a key-role is plaid by the nonlinearity parameter, K, because above a given threshold value, incoherent behavior of the oscillator population undergoes a transition to a (possible partially) synchronized state. Such a parameter is also responsible for various bifurcations, and far away certain critical values corresponding to bifurcating points, a larger number of harmonics is required. Finally in Section 8, the main points of the paper are summarized.

1. Statement of the problem

Consider the following problem, for $(\theta, t, \omega) \in [0, 2\pi] \times [0, T] \times [-L, L]$,

$$\frac{\partial \rho}{\partial t} = D\,\frac{\partial^2 \rho}{\partial \theta^2} - \omega\,\frac{\partial \rho}{\partial \theta} - K\,\frac{\partial}{\partial \theta}\left[\rho \int_0^{2\pi}\int_{-\infty}^{+\infty} \sin(\varphi - \theta)\,\rho(\varphi, t, \omega)\,g(\omega)\,\mathrm{d}\omega\,\mathrm{d}\varphi\right], \quad (1.1)$$

$$\rho(\theta, 0, \omega) = \rho^0(\theta, \omega), \quad (1.2)$$

$$\rho\,|_{\theta=0} = \rho\,|_{\theta=2\pi}, \qquad \left.\frac{\partial \rho}{\partial \theta}\right|_{\theta=0} = \left.\frac{\partial \rho}{\partial \theta}\right|_{\theta=2\pi}. \quad (1.3)$$

Throughout the paper, we assume the following properties for the data, cf. Lavrentiev & Spigler (2000).

CONDITIONS I The initial value of the probability density, $\rho^0(\theta, \omega)$, is supposed to be: ($\mathbf{a_1}$) 2π–periodic in θ; ($\mathbf{a_2}$) smooth, say $\rho^0 \in C^\infty$ in both variables, for simplicity (this implies, in particular, that ρ^0 will be uniformly bounded with respect to the frequency parameter, ω); ($\mathbf{a_3}$) positive,

$$\rho^0(\theta, \omega) > 0; \quad (1.4)$$

and ($\mathbf{a_4}$) normalized,

$$\int_0^{2\pi} \rho^0(\theta, \omega)\,\mathrm{d}\theta = 1. \quad (1.5)$$

The frequency distribution density, $g(\omega)$, is assumed to be: ($\mathbf{b_1}$) nonnegative, $g(\omega) \geqslant 0$; ($\mathbf{b_2}$) integrable, $g \in L^1(\mathbf{R})$; and ($\mathbf{b_3}$) compactly supported, $\operatorname{supp} g \subset [-L, L]$, for some $L > 0$.

It is also understood that the ‘free parameter’ ω in (1.1) must be picked up from $\operatorname{supp} g$.

Consider the Fourier series representation of the solution $\rho(\theta, t, \omega)$ to (1.1)–(1.5),

$$\rho(\theta, t, \omega) = \sum_{n=-\infty}^{\infty} \rho_n(t, \omega)\mathrm{e}^{\mathrm{i}n\theta}, \quad (1.6)$$

where

$$\rho_n(t, \omega) = \frac{1}{2\pi}\int_0^{2\pi} \rho\,\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta. \quad (1.7)$$

The functions $\rho_n \equiv \rho_n(t,\omega)$ in (1.7) can be viewed as the solution to the system of infinitely many ordinary differential equations,

$$\dot{\rho}_n = -n^2 D\,\rho_n - \mathrm{i}n\omega\,\rho_n - n\frac{K}{2}\left[\rho_{n+1}\int_{-\infty}^{+\infty}\rho_{-1}g\,\mathrm{d}\omega - \rho_{n-1}\int_{-\infty}^{+\infty}\rho_1 g\,\mathrm{d}\omega\right],$$
$$n = 0, \pm 1, \pm 2, \ldots, \tag{1.8}$$

a dot on ρ_n denoting differentiation with respect to time. The coefficients $\rho_n(t,\omega)$ are *complex-valued*, cf. (1.6) and (1.7), while ρ is *real-valued*. Thus, $\rho_n(t,\omega) = \bar{\rho}_{-n}(t,\omega)$, $n = 0, 1, 2, \ldots$, a bar denoting complex conjugate, and hence, for every $n \in \mathbf{N}$, $\rho_n\rho_{-n} = |\rho_n|^2$.

It is well known that, for every function ρ, smooth in θ, say $\rho \in C^k$, its Fourier coefficient, ρ_n in (1.7), possess a rate of decay given by

$$|\rho_n| \leqslant \frac{M}{n^k}, \tag{1.9}$$

where M does not depend on n. The main purpose of this paper is obtaining bounds for the coefficients M, corresponding to the solution ρ of problem (1.1)–(1.3). For this we use Parseval relation

$$\left\|\frac{\partial^p\rho}{\partial\theta^p}\right\|_{L^2}^2 = 4\pi\sum_{n=1}^{\infty} n^{2p}\,|\rho_n|^2. \tag{1.10}$$

Therefore, whenever an estimate like

$$\int_0^{2\pi}\left(\frac{\partial^p\rho}{\partial\theta^p}\right)^2\mathrm{d}\theta \leqslant C_p \tag{1.11}$$

is available for some positive integer p, we conclude that, for every integer $n \geqslant 1$,

$$|\rho_n(t,\omega)| \leqslant \frac{M_p}{n^p}, \qquad M_p := \frac{\sqrt{C_p}}{2\sqrt{\pi}}, \tag{1.12}$$

M_p, C_p being independent of n. Therefore, retaining only N harmonics to approximate ρ (i.e., truncating the sum in (1.6) from $-N$ to N), the global *error*, ε_N, can be estimated by

$$\|\varepsilon_N(\theta,t,\omega)\|_{L^2(0,2\pi)} \leqslant \frac{\sqrt{C_p}}{(N+1)^p}, \tag{1.13}$$

valid for every $p \geqslant 1$. Such a bound for the global error can be improved with the help of 'background (*a posteriori*) error analysis'. Suppose we know numerical (computed) values of the first N harmonics coefficients, $\tilde{\rho}_1, \ldots, \tilde{\rho}_N$. As follows from (1.10) and (1.11) for the exact solution coefficients, ρ_n, the inequalities

$$|\rho_n| \leqslant \frac{1}{n^p}\left(\frac{C_p}{4\pi} - \sum_{j=1,\, j\neq n}^{N} j^{2p}|\rho_j|^2\right)^{1/2}, \qquad n \in \{1, \ldots, N\} \tag{1.14}$$

hold. In case the approximation method guarantees the accuracy

$$(1-\gamma)\,|\tilde{\rho}_n| \leqslant |\rho_n| \leqslant (1+\gamma)\,|\tilde{\rho}_n|, \qquad \text{for} \quad \text{some} \quad 0<\gamma<1,$$

we have from (1.14) the estimate

$$|\rho_n| \leqslant \frac{1}{n^p}\left(\frac{C_p}{4\pi} - (1-\gamma)\sum_{j=1,\, j\neq n}^{N} j^{2p}|\tilde{\rho}_j|^2\right)^{1/2}, \qquad n\in\{1,\ldots,N\}. \tag{1.15}$$

A similar argument can be applied to the global error. Indeed, in view of (1.10) and (1.11), the following relations hold:

$$\|\varepsilon_N(\theta,t,\omega)\|^2_{L^2(0,2\pi)} := 4\pi\sum_{n=N+1}^{\infty}|\rho_n|^2 \leqslant \frac{4\pi}{(N+1)^{2p}}\sum_{n=N+1}^{\infty} n^{2p}\,|\rho_n|^2$$

$$= \frac{4\pi}{(N+1)^{2p}}\left[\sum_{n=1}^{\infty} n^{2p}\,|\rho_n|^2 - \sum_{n=1}^{N} n^{2p}\,|\rho_n|^2\right] \leqslant \frac{4\pi}{(N+1)^{2p}}\left[\frac{C_p}{4\pi} - \sum_{n=1}^{N} n^{2p}\,|\rho_n|^2\right].$$

Therefore, we obtain for the global error (cf. (1.15))

$$\|\varepsilon_N(\theta,t,\omega)\|_{L^2(0,2\pi)} \leqslant \frac{f(N,p)}{(N+1)^p} := \frac{2\sqrt{\pi}}{(N+1)^p}\left(\frac{C_p}{4\pi} - (1-\gamma)\sum_{n=1}^{N} n^{2p}\,|\tilde{\rho}_n|^2\right)^{1/2}. \tag{1.16}$$

REMARK 1.1 It is possible to derive a pointwise (or even uniform) estimate for the global error of the form $|\varepsilon_N(\theta,t,\omega)| \leqslant f(N,p)/(N+1)^{p-1}$ (cf. Abramowitz & Stegun (1965)).

Below, we establish a number of estimates of the type in (1.11), for several values of the exponents p ($p = 1, 2, 3, \ldots$), with computable values for the constants C_p.

2. L^2 estimates for the solution ρ

In this section we obtain certain relations which will be useful in the sequel. First of all recall from Lavrentiev & Spigler (2000), that the solution to problem (1.1)–(1.3) remains *positive* and *normalized*,

$$\rho(\theta,t,\omega) > 0, \qquad \int_0^{2\pi}\rho(\theta,t,\omega)\,\mathrm{d}\theta = 1, \tag{2.1}$$

for all times (cf. Lavrentiev & Spigler (2000)). Setting

$$A := \int_{-\infty}^{+\infty} g(\omega)\,\mathrm{d}\omega, \tag{2.2}$$

$$\mathcal{K}_c := \int_{-\infty}^{+\infty}\int_0^{2\pi}\cos(\varphi-\theta)\,\rho(\varphi,t,\omega)g(\omega)\,\mathrm{d}\varphi\,\mathrm{d}\omega,$$

$$\mathcal{K}_s := \int_{-\infty}^{+\infty}\int_0^{2\pi}\sin(\varphi-\theta)\,\rho(\varphi,t,\omega)g(\omega)\,\mathrm{d}\varphi\,\mathrm{d}\omega, \tag{2.3}$$

it is easy to see that, in view of (2.1),

$$\sup_{\theta,t} |\mathcal{K}_c| \leqslant A, \quad \sup_{\theta,t} |\mathcal{K}_s| \leqslant A. \tag{2.4}$$

Taking into account (2.4), we obtain by multiplying both sides of equation (1.1) by ρ, integrating with respect to θ, and integrating by parts (Lavrentiev & Spigler, 2000)

$$\frac{\partial}{\partial t}\int_0^{2\pi} \rho^2 \, \mathrm{d}\theta + 2D\int_0^{2\pi} \rho_\theta^2 \, \mathrm{d}\theta = K\int_0^{2\pi} \rho^2 \, \mathcal{K}_c \, \mathrm{d}\theta \leqslant KA\int_0^{2\pi} \rho^2 \, \mathrm{d}\theta. \tag{2.5}$$

REMARK 2.1 Note, at this point, that for $K > 0$ with $KA < 2D$ we can establish that $\|\rho\|_{L^2(0,2\pi)} + \|\rho_\theta\|_{L^2(0,2\pi)}$ is bounded, uniformly in K (for $0 < K < 2D/A$). Moreover

$$\int_0^{2\pi} \rho^2 \, \mathrm{d}\theta \leqslant \frac{1}{2\pi} + \int_0^{2\pi} \rho_\theta^2 \, \mathrm{d}\theta$$

(see, for example, Lavrentiev & Spigler 2000; Mitrinović 1970; Mitrinović *et al.* 1991), and hence, from (2.5) we deduce that

$$\frac{\partial}{\partial t}\int_0^{2\pi} \rho^2 \, \mathrm{d}\theta + (2D - KA)\int_0^{2\pi} \rho_\theta^2 \, \mathrm{d}\theta \leqslant \frac{KA}{2\pi}.$$

Below, we shall confine ourselves to the 'nasty' case of 'large' K's, $KA > 2D$, i.e., of strong nonlinearities and/or small diffusion.

In the sequel, we shall often use the following corollary of the properties of the solution, $\rho(\theta, t, \omega)$, reported in (2.1),

$$\begin{aligned}\int_0^{2\pi} \rho^2 \, \mathrm{d}\theta &\leqslant \int_0^{2\pi} \rho\left[\frac{1}{2\pi} + \int_0^{2\pi} |\rho_\theta| \, \mathrm{d}\theta\right] \mathrm{d}\theta \\ &= \frac{1}{2\pi} + \int_0^{2\pi} |\rho_\theta| \, \mathrm{d}\theta \leqslant \frac{1}{2\pi} + \frac{\pi}{\varepsilon} + \frac{\varepsilon}{2}\int_0^{2\pi} \rho_\theta^2 \, \mathrm{d}\theta,\end{aligned} \tag{2.6}$$

which holds for all positive $\varepsilon > 0$. Setting $\varepsilon = \frac{2D}{KA}$, from (2.5) and (2.6) we conclude that

$$\frac{\partial}{\partial t}\int_0^{2\pi} \rho^2 \, \mathrm{d}\theta + D\int_0^{2\pi} \rho_\theta^2 \, \mathrm{d}\theta \leqslant \frac{KA}{2\pi} + \pi\frac{(KA)^2}{2D}. \tag{2.7}$$

From (2.7) we observe that, for all $t > 0$ such that

$$\frac{\partial}{\partial t}\int_0^{2\pi} \rho^2 \, \mathrm{d}\theta \geqslant 0, \tag{2.8}$$

the inequality

$$\int_0^{2\pi} \rho_\theta^2 \, \mathrm{d}\theta \leqslant \frac{KA}{2\pi D} + \frac{\pi(KA)^2}{2D^2} \tag{2.9}$$

holds.

Using the fact that, for every positive value of the constants a and b,

$$\min_{x>0}\left(\frac{a}{x}+bx\right)=2\sqrt{ab}, \tag{2.10}$$

and that such minimum value is attained for $x=\sqrt{a/b}$, for t such that (2.8) holds, let us use estimate (2.9) in the right side of (2.6) and minimize such right side, in ε, with the help of (2.10). Setting

$$\kappa := \frac{KA}{2D}, \tag{2.11}$$

we obtain that

$$\int_0^{2\pi}\rho^2\,\mathrm{d}\theta \leqslant \frac{1}{2\pi}+2\pi\kappa\left(1+\frac{1}{2\pi^2\kappa}\right)^{1/2},$$

that is

$$\int_0^{2\pi}\rho^2\,\mathrm{d}\theta \leqslant \frac{1}{2\pi}+2\pi\kappa\,\alpha_0 \tag{2.12}$$

with

$$\alpha_0 := \left(1+\frac{1}{2\pi^2\kappa}\right)^{1/2}. \tag{2.13}$$

In view of (2.8), inequality (2.12) yields, for *all* $t>0$, the estimate

$$\|\rho\|_{L^2(0,2\pi)}=\int_0^{2\pi}\rho^2\,\mathrm{d}\theta \leqslant C_0 \tag{2.14}$$

with the right-hand side given by

$$C_0=\max\left\{\int_0^{2\pi}(\rho^0)^2\,\mathrm{d}\theta;\ \frac{1}{2\pi}+2\pi\kappa\alpha_0\right\}. \tag{2.15}$$

3. Estimating the first space derivative of the solution

The next integral inequality can be obtained by multiplying both sides of Eqn (1.1) by $\rho_{\theta\theta}$, integrating with respect to θ (we omit technical details, which can be found in Lavrentiev & Spigler 2000), and applying relation (2.4):

$$\frac{\partial}{\partial t}\int_0^{2\pi}\rho_\theta^2\,\mathrm{d}\theta+2D\int_0^{2\pi}\rho_{\theta\theta}^2\,\mathrm{d}\theta \leqslant KA\int_0^{2\pi}(3\rho_\theta^2+\rho^2)\,\mathrm{d}\theta. \tag{3.1}$$

Multiplying both sides of relation (2.5) by the constant $\alpha\kappa$ with a certain positive α, to be chosen later, and summing side by side to relation (3.1), we conclude that the 'energy-like' functional

$$R_1(t,\omega) := \int_0^{2\pi}[\alpha\kappa\,\rho^2+\rho_\theta^2]\,\mathrm{d}\theta \tag{3.2}$$

satisfies the inequality

$$\frac{\partial R_1(t,\omega)}{\partial t} + \alpha K A \int_0^{2\pi} \rho_\theta^2 \,\mathrm{d}\theta + 2D \int_0^{2\pi} \rho_{\theta\theta}^2 \,\mathrm{d}\theta \leqslant 3KA \int_0^{2\pi} \rho_\theta^2 \,\mathrm{d}\theta + KA(\alpha\kappa + 1) \int_0^{2\pi} \rho^2 \,\mathrm{d}\theta. \tag{3.3}$$

As is well known (see, for example, Mitrinović 1970; Mitrinović *et al.* 1991), any function $f(\theta)$, with $f \in C^2([0, 2\pi])$, for any positive ε satisfies the inequality

$$\int_0^{2\pi} (f'(\theta))^2 \,\mathrm{d}\theta \leqslant \varepsilon \int_0^{2\pi} (f''(\theta))^2 \,\mathrm{d}\theta + \left(\frac{1}{\varepsilon} + \frac{3}{\pi^2}\right) \int_0^{2\pi} f^2(\theta) \,\mathrm{d}\theta. \tag{3.4}$$

Using (3.4) for $f = \rho$ with $\varepsilon = \frac{2D}{3KA} = (3\kappa)^{-1}$ (see (2.11)), along with (2.14), from (3.3) we get

$$\frac{\partial R_1(t,\omega)}{\partial t} + \alpha K A \int_0^{2\pi} \rho_\theta^2 \,\mathrm{d}\theta \leqslant KA\left[(1 + \alpha\kappa) + 9\kappa + \frac{9}{\pi^2}\right] C_0. \tag{3.5}$$

Consider, again (see inequalities (2.8)–(2.12)), values of $t > 0$ such that

$$\frac{\partial R_1(t,\omega)}{\partial t} \geqslant 0. \tag{3.6}$$

For such ts, relation (3.5) yields

$$\int_0^{2\pi} \rho_\theta^2 \,\mathrm{d}\theta \leqslant C_0 \left[\kappa + \frac{9\kappa + (9/\pi^2) + 1}{\alpha}\right], \tag{3.7}$$

and, consequently, in case (3.6) holds, we have from (3.2) (using also (2.14)),

$$R_1(t,\omega) \leqslant \kappa C_0 + C_0 \frac{9\kappa + (9/\pi^2) + 1}{\alpha} + \kappa\alpha C_0. \tag{3.8}$$

Choosing for the parameter α the value which minimizes the right-hand side of (3.8), we conclude that, for *all* $t > 0$, the inequality

$$\int_0^{2\pi} \rho_\theta^2 \,\mathrm{d}\theta \leqslant R_1 \leqslant C_1 \tag{3.9}$$

holds, with

$$C_1 := \max\left\{\int_0^{2\pi} [\alpha_1\kappa(\rho^0)^2 + (\rho_\theta^0)^2];\ \kappa C_0 (1 + 2\alpha_1)\right\}, \tag{3.10}$$

where

$$\alpha_1 := 3\left(1 + \frac{1}{\pi^2\kappa} + \frac{1}{9\kappa}\right)^{1/2}. \tag{3.11}$$

4. Recurrence formulae

In this section, we give the background to obtaining estimates like those in (1.11), for any given positive integer p, through a recurrence algorithm. The basic technique rests on energy-like estimates as it was in Lavrentiev & Spigler (2000).

First of all, we observe that the functions $\mathcal{K}_c$ and $\mathcal{K}_s$ defined in (2.3) have the following properties:

$$\mathcal{K}_c'(\theta, t) = \mathcal{K}_s(\theta, t), \qquad \mathcal{K}_s'(\theta, t) = -\mathcal{K}_c(\theta, t), \tag{4.1}$$

where a *prime* denotes differentiation with respect to θ. In the sequel we shall use for short the notation

$$\rho^{(n)} := \frac{\partial^n \rho}{\partial \theta^n} \tag{4.2}$$

for the nth partial derivative of the solution $\rho(\theta, t, \omega)$ with respect to θ.

The following obvious identities hold for any positive integer p:

$$\int_0^{2\pi} \rho_t\, \rho^{(2p)}\, \mathrm{d}\theta = (-1)^p \frac{1}{2} \frac{\partial}{\partial t} \int_0^{2\pi} (\rho^{(p)})^2\, \mathrm{d}\theta, \tag{4.3}$$

$$-\omega \int_0^{2\pi} \rho^{(1)}\, \rho^{(2p)}\, \mathrm{d}\theta = 0, \tag{4.4}$$

$$D \int_0^{2\pi} \rho^{(2)}\, \rho^{(2p)}\, \mathrm{d}\theta = (-1)^{p-1} D \int_0^{2\pi} (\rho^{(p+1)})^2\, \mathrm{d}\theta. \tag{4.5}$$

The first two relations can be obtained by p integrations by parts, the third by $p-1$ integrations by parts, and using the smoothness of ρ.

In view of (4.3)–(4.5), multiplying Eqn (1.1) by $\rho^{(2p)}$ and integrating with respect to θ, we obtain

$$\frac{\partial}{\partial t} \int_0^{2\pi} (\rho^{(p)})^2\, \mathrm{d}\theta + 2D \int_0^{2\pi} (\rho^{(p+1)})^2\, \mathrm{d}\theta = (-1)^p\, 2\, K\, \Gamma^p(\rho), \tag{4.6}$$

where $\Gamma^p(\rho)$ stands for the integral

$$\Gamma^p(\rho) := -\int_0^{2\pi} (\rho^{(1)}\, \mathcal{K}_s - \rho\, \mathcal{K}_c)\, \rho^{(2p)}\, \mathrm{d}\theta = \int_0^{2\pi} \rho \mathcal{K}_s \rho^{(2p+1)} d\theta. \tag{4.7}$$

We now obtain some basic recursions. Let us write the integrals $\Gamma^p(\rho)$ defined in (4.7) as

$$\Gamma^p(\rho) = \Gamma_s^p(\rho) + \Gamma_c^p(\rho), \tag{4.8}$$

where

$$\Gamma_s^p(\rho) := -\int_0^{2\pi} \rho^{(1)}\, \mathcal{K}_s \rho^{(2p)}\, \mathrm{d}\theta, \qquad \Gamma_c^p(\rho) := \int_0^{2\pi} \rho\, \mathcal{K}_c \rho^{(2p)}\, \mathrm{d}\theta. \tag{4.9}$$

Then we obtain

$$\begin{aligned}\Gamma_s^p(\rho) &= -\int_0^{2\pi} \rho^{(1)}\,\mathcal{K}_s \rho^{(2p)}\,\mathrm{d}\theta = \int_0^{2\pi} (\rho')^{(1)}\,\mathcal{K}_s\,(\rho')^{(2p-2)}\,\mathrm{d}\theta - \int_0^{2\pi} \rho'\,\mathcal{K}_c\,(\rho')^{(2p-2)}\,\mathrm{d}\theta \\ &= -\Gamma_s^{p-1}(\rho') - \Gamma_c^{p-1}(\rho');\end{aligned} \tag{4.10}$$

$$\begin{aligned}\Gamma_c^p(\rho) &= \int_0^{2\pi} \rho\,\mathcal{K}_c \rho^{(2p)}\,\mathrm{d}\theta = -\int_0^{2\pi} \rho'\,\mathcal{K}_c\,(\rho')^{(2p-2)}\,\mathrm{d}\theta - \int_0^{2\pi} \rho\,\mathcal{K}_s\,\rho^{(2p-1)}\,\mathrm{d}\theta \\ &= -\Gamma_c^{p-1}(\rho') + \int_0^{2\pi} \rho'\,\mathcal{K}_s\,\rho^{(2p-2)}\,\mathrm{d}\theta - \int_0^{2\pi} \rho\,\mathcal{K}_c\,\rho^{(2p-2)}\,\mathrm{d}\theta \\ &= -\Gamma_c^{p-1}(\rho') - \Gamma_s^{p-1}(\rho) - \Gamma_c^{p-1}(\rho).\end{aligned} \tag{4.11}$$

Now, being

$$\Gamma_s^1(\rho) := -\int_0^{2\pi} \rho^{(1)}\,\mathcal{K}_s\,\rho^{(2)}\,\mathrm{d}\theta = -\frac{1}{2}\int_0^{2\pi} (\rho^{(1)})^2\,\mathcal{K}_c\,\mathrm{d}\theta \tag{4.12}$$

and

$$\begin{aligned}\Gamma_c^1(\rho) &:= \int_0^{2\pi} \rho\,\mathcal{K}_c \rho^{(2)}\,\mathrm{d}\theta = -\int_0^{2\pi} (\rho^{(1)})^2\,\mathcal{K}_c\,\mathrm{d}\theta - \int_0^{2\pi} \rho\rho^{(1)}\,\mathcal{K}_s \\ &= -\frac{1}{2}\int_0^{2\pi} [2(\rho^{(1)})^2 + \rho^2]\,\mathcal{K}_c\,\mathrm{d}\theta,\end{aligned} \tag{4.13}$$

the recursions to calculate integrals, $\Gamma^p(\rho)$ could be derived. Indeed, estimating Γ^p as

$$\Gamma^p(\rho) \leqslant A \sum_{j=0}^{p} B_j^p \int_0^{2\pi} (\rho^{(j)})^2\,\mathrm{d}\theta$$

and making use of the recursions (4.10) and (4.11), we shall come up with the following relations on coefficients B_j^p:

$$B_j^p = s_j^p + c_j^p, \quad p = 1, 2, \ldots, \quad j = 0, 1, 2, \ldots, p, \tag{4.14}$$

$$s_j^p = s_{j-1}^{p-1} + c_{j-1}^{p-1}, \tag{4.15}$$

$$c_j^p = c_{j-1}^{p-1} + s_j^{p-1} + c_j^{p-1}, \quad p = 2, 3, \ldots, \quad j = 1, 2, \ldots, p. \tag{4.16}$$

When $j > p$, we set

$$s_j^p = c_j^p = 0, \tag{4.17}$$

and the initial values are

$$s_0^1 = 0, \quad c_0^1 = 1, \quad s_1^1 = 1, \quad c_1^1 = 2. \tag{4.18}$$

Moreover,

$$s_0^p = 0, \qquad c_0^p = 1 \quad p = 1, 2, 3, \dots. \tag{4.19}$$

Here above relations (4.15) follow from (4.10), while formulae (4.16) follow from equalities (4.11). All these recursions can be summed up as a theorem.

THEOREM 4.1 For every positive integer p, the solution $\rho(\theta, t, \omega)$ to problems (1.1)–(1.3) satisfies the inequality

$$\frac{\partial}{\partial t} \int_0^{2\pi} (\rho^{(p)})^2 \, \mathrm{d}\theta + 2D \int_0^{2\pi} (\rho^{(p+1)})^2 \, \mathrm{d}\theta \leqslant KA \sum_{j=0}^{p} B_j^p \int_0^{2\pi} (\rho^{(j)})^2 \, \mathrm{d}\theta, \tag{4.20}$$

where the constants B_j^p can be evaluated recursively from the relation

$$B_j^p = B_j^{p-1} + 2\, B_{j-1}^{p-1} - B_{j-2}^{p-2}, \tag{4.21}$$

where $B_j^p = 0$ when $j > p$ and $j < 0$. Note also that $B_0^p = 1$, and that all the coefficients B_j^p are positive, cf. (4.14)–(4.19).

Some properties of the coefficients B_j^p as well as their sums are here given. From (4.14)–(4.18) we have

$$B_p^p := s_p^p + c_p^p = s_{p-1}^{p-1} + 2c_{p-1}^{p-1},$$

and from (4.16) and (4.18)

$$c_p^p = 2,$$

and

$$s_p^p = s_{p-1}^{p-1} + c_{p-1}^{p-1} = s_1^1 + 2(p-1) = 2p - 1, \qquad \text{for} \quad p = 1, 2, \dots$$

Therefore,

$$B_p^p = 2p + 1, \quad p = 1, 2, \dots \tag{4.22}$$

In a similar way it is easy to see that

$$B_1^p = 2p + 1, \quad p = 1, 2, \dots \tag{4.23}$$

5. Estimating norms of solution ρ

We first derive a basic integral inequality. Consider the inequality in (4.20) along with the same with $p - 1$ replacing p,

$$\begin{aligned} &\frac{\partial}{\partial t} \int_0^{2\pi} (\rho^{(p-1)})^2 \, \mathrm{d}\theta + 2D \int_0^{2\pi} (\rho^{(p)})^2 \, \mathrm{d}\theta \\ &\leqslant KA\, B_{p-1}^{p-1} \int_0^{2\pi} (\rho^{(p-1)})^2 \, \mathrm{d}\theta + KA \sum_{j=0}^{p-2} B_j^{p-1} \int_0^{2\pi} (\rho^{(j)})^2 \, \mathrm{d}\theta, \end{aligned} \tag{5.1}$$

$$\begin{aligned}&\frac{\partial}{\partial t}\int_0^{2\pi}(\rho^{(p)})^2\,\mathrm{d}\theta+2D\int_0^{2\pi}(\rho^{(p+1)})^2\,\mathrm{d}\theta\\&\leqslant KA\,B_p^p\int_0^{2\pi}(\rho^{(p)})^2\,\mathrm{d}\theta+KA\sum_{j=0}^{p-1}B_j^p\int_0^{2\pi}(\rho^{(j)})^2\,\mathrm{d}\theta.\end{aligned}\tag{5.2}$$

Multiplying both sides of (5.1) by $\alpha\kappa$ for a certain positive α (to be chosen later), and summing up, side to side, to (5.2), we obtain for the integral

$$R_p(t,\omega):=\int_0^{2\pi}[\alpha\kappa\,(\rho^{(p-1)})^2+(\rho^{(p)})^2]\,\mathrm{d}\theta,\tag{5.3}$$

$$\begin{aligned}&\frac{\partial R_p}{\partial t}+\alpha KA\int_0^{2\pi}(\rho^{(p)})^2\,\mathrm{d}\theta+2D\int_0^{2\pi}(\rho^{(p+1)})^2\,\mathrm{d}\theta\\&\leqslant KA\,B_p^p\int_0^{2\pi}(\rho^{(p)})^2\,\mathrm{d}\theta+KA\sum_{j=0}^{p-1}[\alpha\kappa\,B_j^{p-1}+B_j^p]\int_0^{2\pi}(\rho^{(j)})^2\,\mathrm{d}\theta.\end{aligned}\tag{5.4}$$

By (1.13) it then follows that

$$\begin{aligned}&\frac{\partial R_p}{\partial t}+\alpha KA\int_0^{2\pi}(\rho^{(p)})^2\,\mathrm{d}\theta+2D\int_0^{2\pi}(\rho^{(p+1)})^2\,\mathrm{d}\theta\\&\leqslant KA\,B_p^p\int_0^{2\pi}(\rho^{(p)})^2\,\mathrm{d}\theta+KA\left[\alpha\kappa\sum_{j=0}^{p-1}B_j^{p-1}\,C_j+\sum_{j=0}^{p-1}B_j^p\,C_j\right]\mathrm{d}\theta.\end{aligned}\tag{5.5}$$

We are ready now to obtain L^2 estimates for higher derivatives of solutions. In view of (4.22) and (4.23), using the estimate in (3.4) with $f=\rho^{(p-1)}$ and $\varepsilon=((2p+1)\kappa)^{-1}$, we deduce from (5.5)

$$\begin{aligned}\frac{\partial R_p}{\partial t}+\alpha KA\int_0^{2\pi}(\rho^{(p)})^2\,\mathrm{d}\theta&\leqslant KA\left[(2p+1)^2\kappa+\frac{3(2p+1)}{\pi^2}\right]\int_0^{2\pi}(\rho^{(p-1)})^2\,\mathrm{d}\theta\\&\quad+KA\left[\alpha\kappa\sum_{j=0}^{p-1}B_j^{p-1}\,C_j+\sum_{j=0}^{p-1}B_j^p\,C_j\right].\end{aligned}\tag{5.6}$$

As in Section 3, in case $\frac{\partial R_p}{\partial t}\geqslant 0$ the latter relation entails

$$\begin{aligned}&\int_0^{2\pi}(\rho^{(p)})^2\,\mathrm{d}\theta\leqslant\kappa\sum_{j=0}^{p-1}B_j^{p-1}\,C_j\\&+\frac{1}{\alpha}\left[(2p+1)^2\kappa+\frac{3(2p+1)}{\pi^2}+B_{p-1}^p\right]C_{p-1}+\frac{1}{\alpha}\sum_{j=0}^{p-2}B_j^p\,C_j,\end{aligned}\tag{5.7}$$

and hence, for values of time such that $\frac{\partial R_p}{\partial t} \geqslant 0$, we obtain from (5.3)

$$R_p(t,\omega) \leqslant \alpha\kappa\, C_{p-1} + \kappa \sum_{j=0}^{p-1} B_j^{p-1}\, C_j$$
$$+ \frac{1}{\alpha}\left[(2p+1)^2\kappa + \frac{3(2p+1)}{\pi^2} + B_{p-1}^p\right] C_{p-1} + \frac{1}{\alpha}\sum_{j=0}^{p-2} B_j^p\, C_j. \tag{5.8}$$

Minimizing the right-hand side of (5.8) with respect to α (see (2.10)), we finally conclude that, in any case (i.e., for all $t > 0$),

$$\int_0^{2\pi} (\rho^{(p)})^2\, \mathrm{d}\theta \leqslant C_p \tag{5.9}$$

with

$$C_p := \max\left\{\int_0^{2\pi} [\alpha_p\kappa((\rho^0)^{(p-1)})^2 + ((\rho^0)^{(p)})^2]\,\mathrm{d}\theta;\ \kappa \sum_{j=0}^{p-1} B_j^{p-1}\, C_j + 2\kappa\, C_{p-1}\,\alpha_p\right\}, \tag{5.10}$$

where

$$\alpha_p := (2p+1)\left[1 + \frac{3}{(2p+1)\pi^2\kappa} + \frac{B_{p-1}^p}{(2p+1)^2\kappa} + \frac{1}{(2p+1)^2\kappa\, C_{p-1}} \sum_{j=0}^{p-2} B_j^p\, C_j\right]^{1/2}. \tag{5.11}$$

Being hard, as usual, to obtain error bounds that are, at the same time, both *computable* and *realistic*, we derive here below explicit estimates for C_p. These are, however, by far overestimates. In fact, we have the following

REMARK 5.1 As, by its own definition in (5.11), $\alpha_p > 2p + 1$, the following bound from below can be derived for the constants C_p:

$$C_p > (2\kappa)^p\,(2p+1)!!\, C_0. \tag{5.12}$$

This inequality is valid under the assumption that the second term on the right-hand side in (5.10) is greater than the first one, which is certainly true for large values of κ.

This shows that C_p must grow very rapidly with p, which in fact yields too poor results in terms of number of harmonics required to achieve a prescribed error bound in (1.16). Recall however that the constant C_p is itself an upper bound, likely overestimated, for the left-hand side of (5.9).

6. The explicit form of some constants C_p and the steady state case

Theorem 4.1 allows us to produce bounds for the constants C_p in (1.13). Indeed, after simple calculations we get

$$C_2 := \max\left\{\int_0^{2\pi}[\alpha_2\kappa(\rho^0_\theta)^2 + (\rho^0_{\theta\theta})^2]\,\mathrm{d}\theta;\ \kappa(2\alpha_2+3)\,C_1 + \kappa\,C_0\right\}, \tag{6.1}$$

$$C_3 := \max\left\{\int_0^{2\pi}[\alpha_3\kappa(\rho^0_{\theta\theta})^2 + (\rho^0_{\theta\theta\theta})^2]\,\mathrm{d}\theta;\ \kappa(5C_2+5C_1+C_0) + 2\kappa\,C_2\,\alpha_3\right\}, \tag{6.2}$$

$$\begin{aligned} C_4 := \max\Bigg\{&\int_0^{2\pi}\left[\alpha_4\kappa\left(\frac{\partial^3\rho^0}{\partial\theta^3}\right)^2 + \left(\frac{\partial^4\rho^0}{\partial\theta^4}\right)^2\right]\mathrm{d}\theta;\\ &\kappa(7\,C_3 + 14\,C_2 + 7\,C_1 + C_0) + 2\kappa\,C_3\,\alpha_4\Bigg\}, \end{aligned} \tag{6.3}$$

and

$$\begin{aligned} C_5 := \max\Bigg\{&\int_0^{2\pi}\left[\alpha_5\kappa\left(\frac{\partial^4\rho^0}{\partial\theta^4}\right)^2 + \left(\frac{\partial^5\rho^0}{\partial\theta^5}\right)^2\right]\mathrm{d}\theta;\\ &\kappa(9\,C_4 + 30\,C_3 + 27\,C_2 + 9\,C_1 + C_0) + 2\kappa\,C_4\,\alpha_5\Bigg\}. \end{aligned} \tag{6.4}$$

Here

$$\alpha_2 := 5\left(1 + \frac{3}{5\kappa\pi^2} + \frac{1}{5\kappa} + \frac{C_0}{25\kappa C_1}\right)^{1/2}, \tag{6.5}$$

$$\alpha_3 := 7\left[1 + \frac{3}{7\pi^2\kappa} + \frac{2}{7\kappa} + \frac{C_1}{7\kappa C_2} + \frac{C_0}{49\kappa C_2}\right]^{1/2}, \tag{6.6}$$

$$\alpha_4 := 9\left[1 + \frac{1}{3\pi^2\kappa} + \frac{10}{27\kappa} + \frac{C_2}{3\kappa\,C_3} + \frac{C_1}{9\kappa C_3} + \frac{C_0}{81\kappa C_3}\right]^{1/2}, \tag{6.7}$$

and

$$\begin{aligned} \alpha_5 &:= 11\left[1 + \frac{3}{11\pi^2\kappa} + \frac{B^5_4}{121\kappa} + \frac{1}{121\kappa\,C_4}\sum_{j=0}^{3} B^5_j\,C_j\right]^{1/2}\\ &= 11\left[1 + \frac{3}{11\pi^2\kappa} + \frac{5}{11\kappa} + \frac{C_0}{121\kappa C_4} + \frac{C_1}{11\kappa C_4} + \frac{4C_2}{11\kappa C_4} + \frac{7C_3}{11\kappa C_4}\right]^{1/2}. \end{aligned} \tag{6.8}$$

It is important in itself to compute the *steady state* solution, say $\rho^s(\theta,\omega)$, which solves the equation

$$D\,\rho^s_{\theta\theta} - \omega\,\rho^s_\theta - K\frac{\partial}{\partial\theta}\left[\rho^s\int_0^{2\pi}\int_{-\infty}^{+\infty}\sin(\varphi-\theta)\,\rho^s(\varphi,\omega)\,g(\omega)\,\mathrm{d}\omega\,\mathrm{d}\varphi\right] = 0, \tag{6.9}$$

plus periodic boundary values. In this case, the constants C_p above, say C_p^s, could be reduced. Indeed, in order to estimate the integral of $(\rho_\theta^s)^2$, we only need the 'stationary version' of (2.5),

$$\int_0^{2\pi} (\rho_\theta^s)^2 \,\mathrm{d}\theta \leqslant \kappa \int_0^{2\pi} (\rho^s)^2 \,\mathrm{d}\theta. \tag{6.10}$$

Using (2.6) with $\varepsilon = \kappa^{-1}$ and (6.10), we get

$$\int_0^{2\pi} (\rho_\theta^s)^2 \,\mathrm{d}\theta \leqslant C_1^s := \kappa \left(\frac{1}{\pi} + 2\pi\kappa \right). \tag{6.11}$$

Similarly, we obtain (see (3.1))

$$\int_0^{2\pi} (\rho_{\theta\theta}^s)^2 \,\mathrm{d}\theta \leqslant \kappa \int_0^{2\pi} [3(\rho_\theta^s)^2 + (\rho^s)^2] \,\mathrm{d}\theta, \tag{6.12}$$

and then, from (6.11),

$$\int_0^{2\pi} (\rho_{\theta\theta}^s)^2 \,\mathrm{d}\theta \leqslant C_2^s := \kappa (3\, C_1^s + C_0^s), \tag{6.13}$$

where C_0^s is given by

$$\int_0^{2\pi} (\rho^s)^2 \,\mathrm{d}\theta \leqslant C_0^s \leqslant \frac{1}{2\pi} + C_1^s,$$

see Remark 2.1. We conclude, by induction, that

$$C_p^s = \kappa \sum_{j=0}^{p-1} B_j^{p-1}\, C_j^s, \tag{6.14}$$

cf. (4.20) and (5.10), therefore

$$C_3^s = \kappa\, (5\, C_2^s + 5\, C_1^s + C_0^s), \tag{6.15}$$
$$C_4^s = \kappa\, (7\, C_3^s + 14\, C_2^s + 7\, C_1^s + C_0^s), \tag{6.16}$$
$$C_5^s = \kappa\, (9\, C_4^s + 30\, C_3^s + 27\, C_2^s + 9\, C_1^s + C_0^s). \tag{6.17}$$

7. Numerical results

Numerical simulations, based on the spectral method developed in the previous sections, have been conducted on the Kuramoto–Sakaguchi integro-differential Eqn (1.1). The numerical solution, approximated with N Fourier harmonic, say $\rho^{(N)}$, favorably agrees with the theoretical estimates obtained using the computed values for the constants C_p in (1.11). As usual, the theoretical bounds by far represent overestimates. Both cases of a discrete unimodal frequency distribution, $g(\omega) = \delta(\omega)$, and that of a Lorentzian profile, $g(\omega) = (\sigma/\pi)/(\sigma^2 + \omega^2)$, where σ measures the spread of the profile, have been considered; both cases are important in modelling oscillator populations. The number of

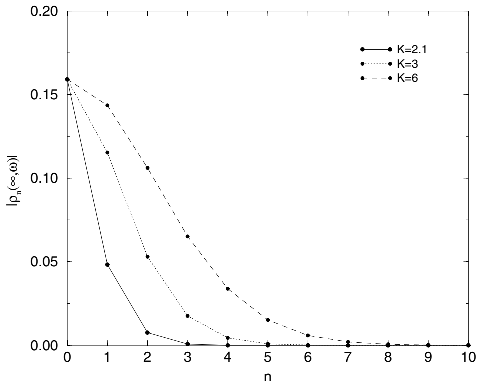

FIG. 1. The coefficients $|\rho_n(\infty, \omega)|$ corresponding to the stationary solution are shown for various coupling strength parameters, K. The frequency distribution here is $g(\omega) = \delta(\omega)$, and the diffusion constant $D = 1$.

harmonics, N is a key parameter, of course, as well as the smoothness parameter, p, and numerical experiments show several cases. The systems of *ordinary* differential equations in (1.8) have been solved by a variable step Runge–Kutta–Fehlberg routine (Press *et al.*, 1996). There are at the present time much better codes for solving (stiff) systems of ordinary differential equations (see, for example, Hairer *et al.* 1993; Hairer & Wanner 1996), but it does not seem so important here to use the best codes, since we stress mostly the *reduction* of the Kuramoto integroparabolic differential equation to a system of ordinary differential equations by a spectral approach, and leave to the user, the choice of the treatment of the ensuing system. As for the truncation of the system of *infinitely* many ordinary differential equations, some experimentation has been made, according to the following strategy: a number N, $2N$, $4N$ of harmonics has been used, and the choice of N was made when the error dropped below a given tolerance (typically 10^{-12}).

A number of experiments were first conducted for the stationary case, and thus the coefficients $|\rho_n(\infty, \omega)|$, corresponding to the limiting values of $\rho_n(t, \omega)$ for $t \to +\infty$, have been computed. In Fig. 1, such coefficients are shown for three different values of the 'nonlinearity parameter' K. The comparison between the theoretical estimates $\sqrt{C_p^s/\pi}/2n^p$, $p = 0, 1, 2$, for $|\rho_n(\infty, \omega)|$, and the numerical simulation are shown in Fig. 2. In Fig. 3, the first three harmonics as functions of the parameter K are depicted. In Fig. 4, $|\rho_n(\infty, \omega)|$ is compared with the theoretical estimates in Eqn (1.15), $p = 1, 2$. In Fig. 5, the results obtained by the present spectral method and by an implicit finite difference method, using forward differences in time and central differences in space (Crank–Nicholson scheme), with step-size $\Delta\theta = 0{\cdot}04$, and $\Delta t = 10^{-4}$, are compared with the analytical solution to the stationary problem being studied. The linear part was treated implicitly, while the nonlinear term was always treated explicitly. To illustrate the efficiency of both numerical methods, we compared the overall CPU time needed to compute the solution. By using the spectral method with $N = 12$ harmonics, the CPU time is approximately 25 times faster than the difference method with $\Delta\theta = 0{\cdot}04$, $\Delta t = 10^{-4}$ step-sizes, using the same parameters.

In Fig. 6, the pointwise error between the stationary solution, $\rho(\theta, \infty, \omega)$, and its

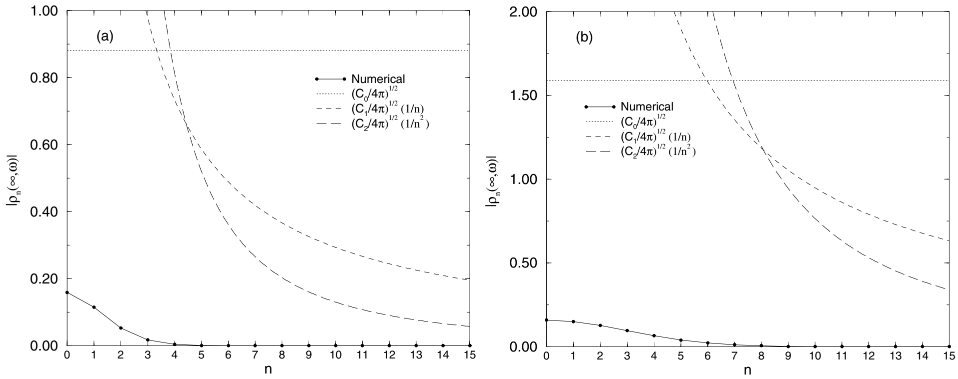

FIG. 2. Comparison between the estimates determined by C_0^s, C_1^s, C_2^s and the numerical simulations, as functions of n, when the stationary state is attained. Parameters are the same as is in Fig. 1, except that now: (a) $K = 3$, and (b) $K = 10$.

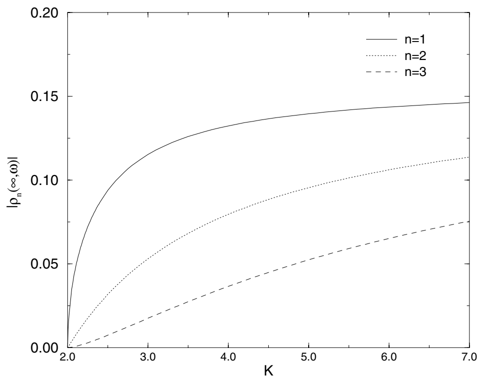

FIG. 3. The first three harmonics, $|\rho_n(\infty, \omega)|$, are shown as functions of the coupling strength, K. The other parameters are as in Fig. 1.

spectral numerical approximation obtained with four harmonics, $\rho^{(4)}(\theta, \infty, \omega)$, is plotted as a function of θ, for $K = 3$ and 6. Note that the theoretical bounds are only a few times larger than the computed values. In Figs 7 and 8 the *global* error, $||\varepsilon_N||$, that one must face when the Fouries series is truncated as in (1.13) and (1.16), is shown along with its estimate, $\sqrt{C_p^s}/(N+1)^p$. This has been carried out for two different frequency distributions: discrete unimodal, in Fig. 7, and Lorentzian profile, for two values of K, in Fig. 8. In Fig. 7(a), the theoretical estimate for the global error is seen to *decrease* increasing p, for N given, despite the asymptotic growth of C_p with p. Numerical quadrature was only needed when a Lorentzian natural frequency distribution or other distributions with 'large' support, i.e., of finite or infinite size (as opposite to the case of delta Dirac distributions), were considered. In this case, the Gauss–Laguerre quadrature rule has been adopted. A reasonable number of nodes were found to be 15.

In the physical literature devoted to populations of nonlinearly coupled oscillators

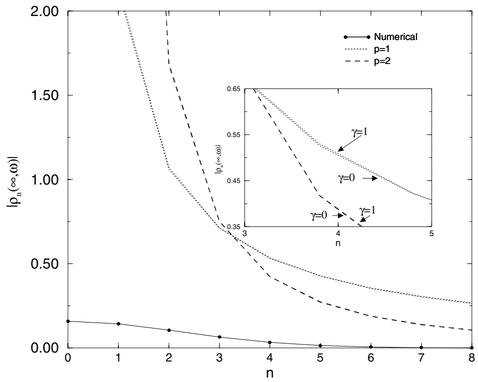

FIG. 4. Comparison between $|\rho_n(\infty, \omega)|$ and the theoretical estimate related to C_1^s and C_2^s. Parameters are as in Fig. 1 with $K = 6$.

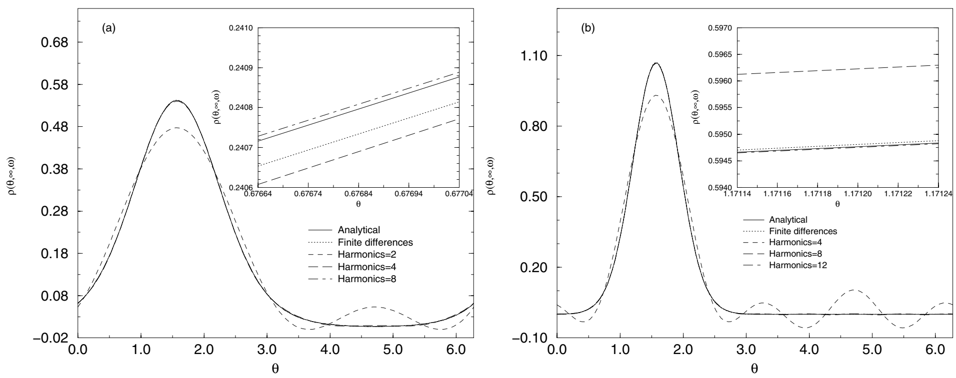

FIG. 5. Comparison among the results of numerical simulations by finite differences with mesh-size $\Delta\theta = 0{\cdot}04$, the spectral method with $N = 2, 4, 8, 12$ harmonics, and the analytical stationary solution. (a) refers to $K = 3$, and (b) to $K = 8$.

subject to mean-field interaction, the so-called (complex-valued) order parameter

$$r(t)\mathrm{e}^{\mathrm{i}\psi(t)} := \int_0^{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\varphi}\, g(\omega)\, \rho(\varphi, \omega, t)\, \mathrm{d}\omega\, \mathrm{d}\varphi, \tag{7.1}$$

is defined, cf. Bonilla *et al.* (1992); Sartoretto *et al.* (1998); Strogatz & Mirollo (1991). Such a quantity, and in particular, the real positive function $r(t)$, is important because $r(t) \equiv 0$ corresponds to the incoherent state and $r(t) \equiv 1$ to the fully synchronized state. In general, $0 \leqslant r(t) \leqslant 1$, and $r(t) \equiv$ const. means partial synchronization. Also, $r(t)$ will be in general, time dependent, while the corresponding distribution function ρ will also depend on θ and ω. Moreover, $r(t)$, which is clearly a kind of macroscopic observable, approaches 1 when $K \to +\infty$. It is easy to see that $r(t)$ can be used to estimate $\mathcal{K}_s$ and $\mathcal{K}_c$, rather then A, cf. (2.2)–(2.4), which suggests to use the *computed values* of r (but actually only that of ρ_1 is needed) to conduct an '*a posteriori* analysis' in the ensuing bounds for C_p. Observations are stated in the following

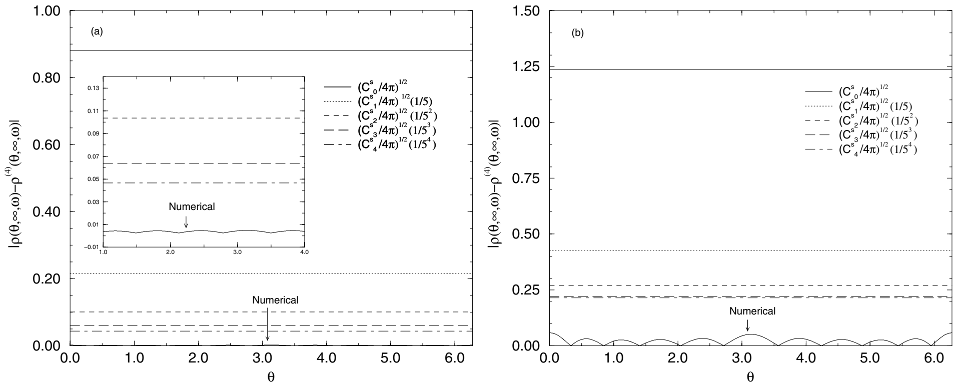

FIG. 6. Comparison between the error made truncating the Fourier series to four harmonics and the analytical estimates for the stationary solution. Here $g(\omega) = \delta(\omega)$, and: (a) $K = 3$, (b) $K = 6$.

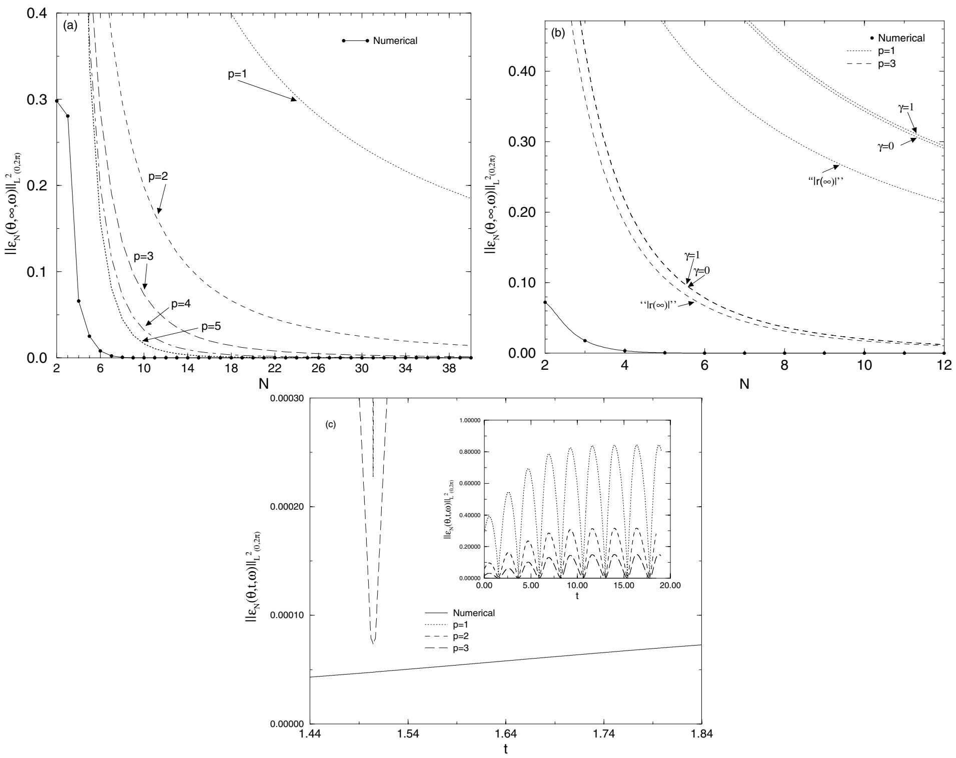

FIG. 7. Global L^2 numerical error when truncating the Fourier series to N harmonics, and analytical estimates as on the right–hand side of (1.16). (a) $\gamma = 1$. The coupling strength is kept fixed to $K = 6$. (b) Comparison between the error estimates with *a posteriori* analysis ($\gamma = 0$, and $|r(t)|$). Here $K = 2{\cdot}3$. (c) Global error and error estimates ($p = 1, 2, 3$), as a funtion of time when using *a posteriori* analysis with $|r(t)|$. The frequency distribution is bimodal with $\omega_0 = 2$, $K = 5$ and $N = 10$ harmonics.

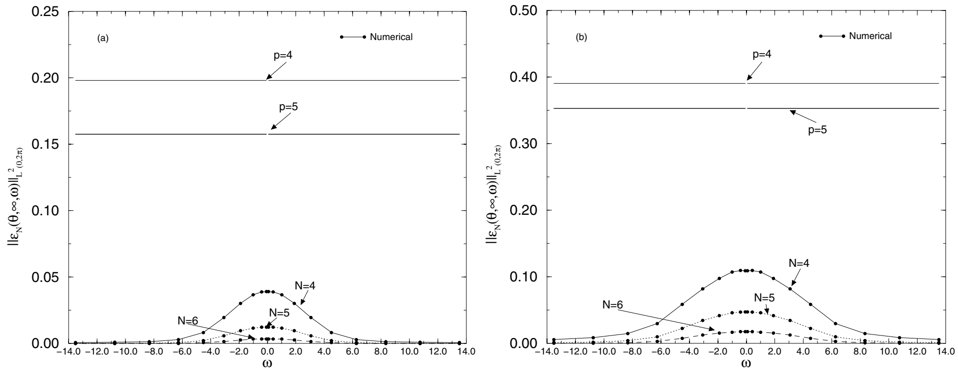

FIG. 8. Global L^2 numerical error in the spectral procedure (retaining $N = 4, 5, 6$ harmonics in the Fourier series), as a function of the frequency, ω, and analytical estimates for the stationary solution, with $p = 4, 5$. The frequency distribution here is a Lorentzian, and: (a) $K = 6$, (b) $K = 8$.

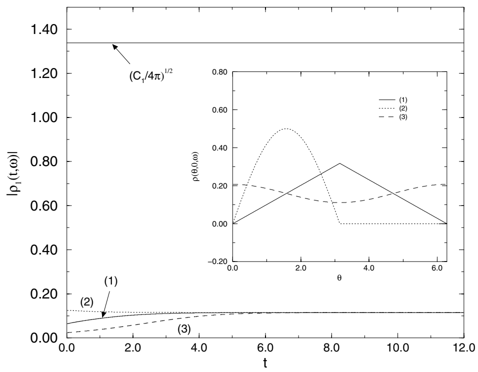

FIG. 9. Time evolution of the first harmonic, $|\rho_1(t, \omega|)$, and comparison with the analytical estimate related to C_1, for three different initial conditions, $\rho^0(\theta, \omega) \equiv \rho(\theta, 0, \omega)$ (see inset). Here $K = 3$ and $g(\omega) = \delta(\omega)$.

REMARK 7.1 Replacing A with $r(t)$ in (2.11) amounts to obtaining *time-dependent estimates*, $C_p \equiv C_p(t)$, where now $\kappa \equiv \kappa(t) := Kr(t)/2D$.

When $\kappa \to 0$ (in both choices, with A or $r(t)$ replacing it), then the estimate 'constants', C_p, reduce to the bounds

$$C_p = \int_0^{2\pi} \left(\frac{\partial^p \rho^0}{\partial \theta^p} \right)^2 \mathrm{d}\theta \tag{7.2}$$

(cf. (6.1)–(6.8)), that is the same estimates obtained for the linear problem (involving the heat equation).

When $\kappa(t)$ is appreciably smaller than 1, then the estimates $C_p(t)$ might be correspondingly improved, with respect to the time independent case.

In Fig. 7(b), it is shown that, in some cases, *a posteriori* analysis may provide rather sharp error estimates, cf. (1.15), (1.16). In Fig. 7(c), the global error is displayed for the

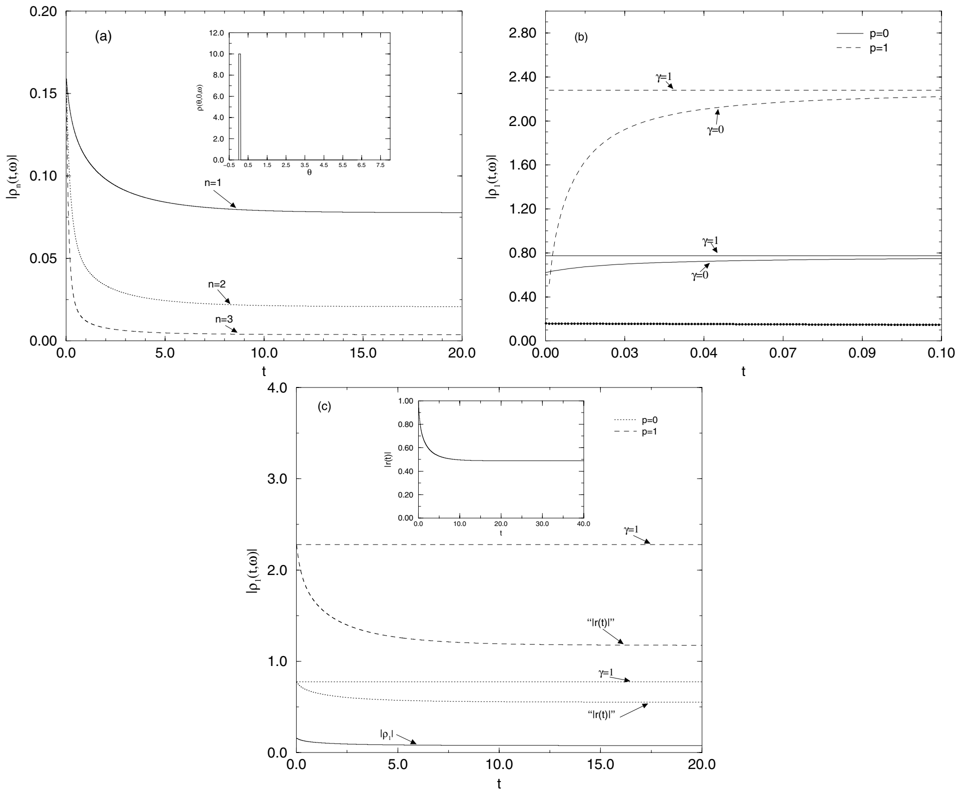

FIG. 10. (a) Time evolution of three different harmonics, $|\rho_n(t, \omega)|$, $n = 1, 2, 3$, for the initial profile shown in the inset. (b), (c) Time evolution of the first harmonic, $n = 1$, compared with the theoretical estimates for $p = 0, 1$. Note that the estimate is improved with the help of the *a posteriori* error analysis ($\gamma = 0$, and $|r(t)|$). Here $K = 2{\cdot}3$ and $g(\omega) = \delta(\omega)$.

case of the *bimodal* frequency distribution, $g(\omega) = [\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]/2$, in which case (for a suitable choice of the parameters), *standing wave* types of solutions exist; in this case, $r(t)$ is indeed time-dependent and oscillatory, and its values are appreciably smaller than 1.

Figures 9–11 refer to the time-dependent case. In Fig. 9, a comparison is made between the analytical estimate corresponding to C_1 and the harmonic $|\rho_1(t, \omega)|$ for three different initial conditions. In Fig. 11, the time dependent solution, ρ, is computed for three different times. Since 80 harmonics have been used for the purpose of accuracy, the FFT algorithm has been used here.

In Fig. 12, the CPU time needed to obtain a given global error between the analytical stationary solution and the numerical solution, is shown as a function of the step-size $\Delta\theta$, when finite differences are used, and as a function of N, when the spectral method is adopted. Polynomial, *vs* spectral convergence is illustrated in Fig. 13(a), where $N = 1/\Delta\theta$ for the finite difference scheme. In any case, the sensitivity of the two methods according to the slopes, and the superior performance of the spectral method is clear. One can see,

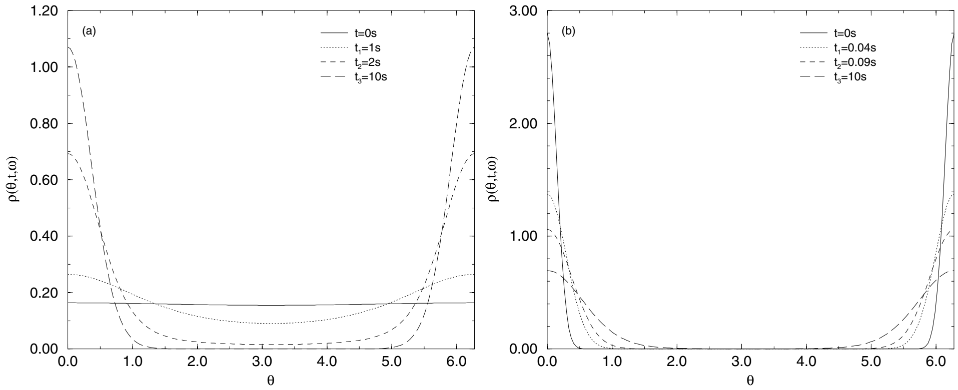

FIG. 11. Time evolution of two different initial profiles: (a) $K = 8$, (b) $K = 4$. Note that $0 < t_1 < t_2 < t_3$. The solution, $\rho(\theta, t, \omega)$, has been computed using the FFT (Fast Fourier Transform) algorithm. Here $g(\omega) = \delta(\omega)$, and $N = 80$.

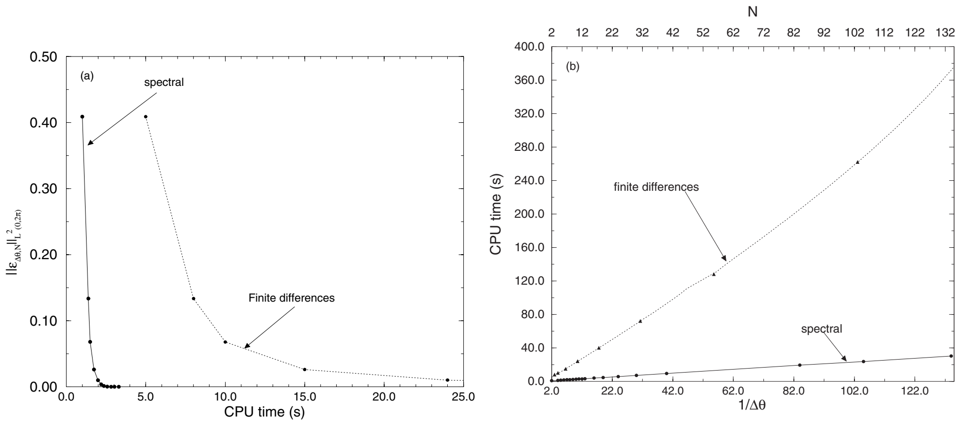

FIG. 12. (a) Comparison of the global error between the analytical stationary solution and the numerical solution obtained by both, finite differences and spectral method, as a function of the CPU time, (b) CPU time as a function of $1/\Delta\theta$ and the number of harmonics N. Here $g(\omega) = \delta(\omega)$, and $K = 8$.

in particular, that stronger nonlinearities (i.e., larger Ks) require higher Ns or smaller $\Delta\theta$s to achieve the same accuracy. In Fig. 13(b), a comparison between finite differences and the spectral approach is made as follows: each point, represents a given fixed value of the global error on the plane $((\Delta\theta)^{-1}, N)$.

8. Summary

A numerical spectral procedure has been introduced to solve typical problems for the Kuramoto (or Kuramoto–Sakaguchi) nonlinear parabolic integrodifferential equation. This equation describes the time evolution of the probability density of populations of infinitely many stochastic phase oscillators, nonlinearly coupled through a ‘mean field’ mechanism.

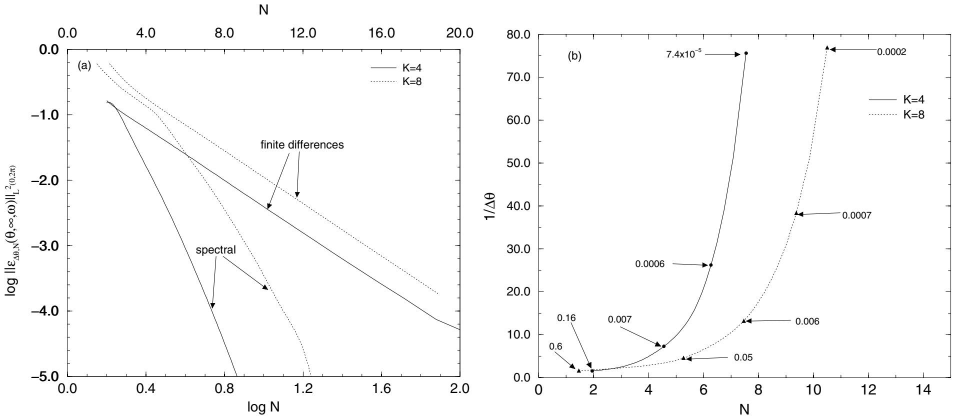

FIG. 13. (a) Log of the global L^2-error as a function of $\log N$, where $N = 1/\Delta\theta$, for the finite difference method, and as a function of N for the spectral method. Here $K = 4$ or 8. (b) Here each point represents a fixed value of the global error (whose value could be read on Fig. (a)) on the plane (N, $1/\Delta\theta$). Some values of the global error are also shown on the figure. Here $g(\omega) = \delta(\omega)$.

Periodicity in space (which is an angle) suggests adopting a Fourier-type spectral method, and estimates for the solution along with its space derivatives of all orders have been obtained. As expected (and usual), such estimates are not very sharp, when compared with the numerically computed quantities. We stress however that they hold uniformly (universally) with respect to any choice of the initial profiles, and of the other physical parameters. Moreover, being the Kuramoto equation the *integro*-differential equation, one might expect a loss of accuracy due to the integral part, at least when the corresponding discretization leads to full matrices. Such a case would be encountered, for instance, when the (given) frequency distribution, $g(\omega)$, is Lorentzian, even though several interesting cases can be modeled by simple delta-Dirac distributions. Several pictures illustrate the performance of the method.

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