Separate Appendix to: "Time Varying Cointegration"

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1 Appendix A: Preliminary Results

1.1 Proof of Lemma 1

Note that the function $\varphi(x)$ is an element of the Hilbert space $L^2[0,1]$ of square integrable real functions on $[0,1]$, with inner product $\langle f,g\rangle = \int_0^1 f(x)g(x)dx$ and associated norm $||f|| = \sqrt{\langle f,f\rangle}$ and metric $||f-g||$.

We first show that the sequence

$$\kappa_j(x) = \begin{cases} 1 & \text{for} \quad j=0, \\ \sqrt{2}\cos(j\pi x) & \text{for} \quad j=1,2,3,.... \end{cases} \tag{1}$$

is a complete orthonormal sequence in $L^2[0,1]$. This result is well-known in the statistics literature (see for example Kronmal and Tarter 1968), but to the best of our knowledge has not been used in the econometrics literature. Therefore, we will prove it here, as follows.

Recall[1] that the functions

$$1,\ \sqrt{2}\cos(i\pi x),\ \sqrt{2}\sin(j\pi x),\ i,j=1,2,3,....,\ x\in[-1,1],$$

form a complete orthonormal sequence in $L^2[-1,1]$ with respect to the uniform density on $[-1,1]$, hence every real function $\psi \in L^2[-1,1]$ satisfies

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \frac{1}{2}\int_{-1}^{1} (\psi(x) - \psi_{m,n}(x))^2 dx \tag{2}$$

[1] See for example Young (1988)

where

$$\psi_{m,n}(x)=\underline{\omega}_0+\sum_{i=1}^{m}\omega_i\sqrt{2}\cos(i\pi x)+\sum_{j=1}^{n}\varpi_j\sqrt{2}\sin(j\pi x) \tag{3}$$

with Fourier coefficients $\underline{\omega}_0=\frac{1}{2}\int_{-1}^{1}\psi(x)\,dx$, $\omega_k=\frac{1}{2}\int_{-1}^{1}\sqrt{2}\cos(k\pi x)\psi(x)\,dx$ and $\varpi_k=\frac{1}{2}\int_{-1}^{1}\sqrt{2}\sin(k\pi x)\psi(x)\,dx$. Now let $\varphi(u)\in L^2[0,1]$ be arbitrary, and let $\psi(x)=\varphi(|x|)$. Then $\psi(x)\in L^2(-1,1)$, with Fourier coefficients

$$\begin{aligned}
\underline{\omega}_0 &= \frac{1}{2}\int_{-1}^{1}\varphi(|x|)\,dx=\int_{0}^{1}\varphi(u)\,du\\
\omega_k &= \frac{1}{2}\int_{-1}^{1}\sqrt{2}\cos(k\pi x)\varphi(|x|)\,dx=\int_{0}^{1}\sqrt{2}\cos(k\pi u)\varphi(u)\,du\\
\varpi_k &= \frac{1}{2}\int_{-1}^{1}\sqrt{2}\sin(k\pi x)\varphi(|x|)\,dx=0
\end{aligned}$$

Hence it follows from (2) and (3) that

$$\begin{aligned}
&\lim_{n\to\infty}\int_{0}^{1}\left(\varphi(u)-\underline{\omega}_0-\sum_{k=1}^{n}\omega_k\sqrt{2}\cos(k\pi u)\right)^2du \qquad (4)\\
&=\frac{1}{2}\lim_{n\to\infty}\int_{-1}^{1}\left(\varphi(|x|)-\underline{\omega}_0-\sum_{k=1}^{n}\omega_k\sqrt{2}\cos(k\pi x)\right)^2dx\\
&=0
\end{aligned}$$

This proves the completeness of (1).

Next, let $t_x=[xT]+1$ for an $x\in[0,1)$, where $[xT]$ is the largest integer $\leq xT$. Then

$$g_m(t_x)=g_m([xT]+1)=\xi_{0,T}+\sqrt{2}\sum_{i=1}^{m}\xi_{i,T}\cos\left[i\pi\left(\frac{[xT]+1}{T}-\frac{1}{2T}\right)\right]$$

where

$$\begin{aligned}
\xi_{0,T} &= \frac{1}{T}\sum_{t=1}^{T}\varphi(t/T)=\int_{0}^{1}\varphi\left(\frac{[yT]+1}{T}\right)dy\\
\xi_{i,T} &= \frac{1}{T}\sum_{t=1}^{T}\varphi(t/T)\sqrt{2}\cos\left[i\pi(t-0.5)/T\right]\\
&= \int_{0}^{1}\varphi\left(\frac{[yT]+1}{T}\right)\sqrt{2}\cos\left[i\pi\left(\frac{[yT]+1}{T}-\frac{1}{2T}\right)\right]dy
\end{aligned}$$

Hence by bounded convergence,

$$\begin{aligned}
\varphi_m(x) &= \lim_{T\to\infty} g_m\left([xT]+1\right) = \xi_0 + \sum_{i=1}^{m} \xi_i \sqrt{2}\cos\left(i\pi x\right), \text{ where} \\
\xi_0 &= \lim_{T\to\infty} \xi_{0,T} = \int_0^1 \varphi\left(y\right) dy, \\
\xi_i &= \lim_{T\to\infty} \xi_{i,T} = \int_0^1 \varphi(y)\sqrt{2}\cos(i\pi y)dy, \; i \geq 1.
\end{aligned}$$

It is now easy to verify that

$$\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T}\left(g\left(t\right) - g_{m,T}\left(t\right)\right)^2 = \int_0^1 \left(\varphi\left(x\right) - \varphi_m(x)\right)^2 dx \tag{5}$$

Note that

$$\int_0^1 \left(\varphi_m(x) - \varphi\left(x\right)\right)^2 dx = \int_0^1 \varphi\left(x\right)^2 dx - \sum_{i=1}^{m} \xi_i^2 > 0,$$

hence $\sum_{i=1}^{\infty} \xi_i^2 \leq \int_0^1 \varphi\left(x\right)^2 dx < \infty$. However, due to the completeness of (1), we also have $\sum_{i=1}^{\infty} \xi_i^2 = \int_0^1 \varphi\left(x\right)^2 dx$. See for example Young (1988, Theorem 4.15, p.37). Thus,

$$\lim_{m\to\infty} \int_0^1 \left(\varphi(x) - \varphi_m(x)\right)^2 dx = 0. \tag{6}$$

Combining (5) and (6), the first part of Lemma 1 follows.

To prove the second part of Lemma 1, suppose that $\varphi(x)$ is q times differentiable, where $q \geq 2$ is *even*, and that $\varphi^{(q)}(x) = d^q\varphi(x)/\left(dx\right)^q$ is square-integrable. Then $\varphi^{(q)}(x) \in L^2\left[0,1\right]$:

$$\lim_{m\to\infty} \int_0^1 \left(\varphi^{(q)}(x) - \sum_{i=1}^{m}(-1)^{q/2}\pi^q i^q \xi_i \sqrt{2}\cos\left(i\pi x\right)\right)^2 dx,$$

where $\int_0^1 \left(\varphi^{(q)}(x)\right)^2 dx = \pi^{2q}\sum_{i=1}^{\infty} i^{2q}\xi_i^2 < \infty$. Now for $m \geq 1$,

$$\begin{aligned}
&\int_0^1 \left(\varphi_m(x) - \varphi\left(x\right)\right)^2 dx = \sum_{i=m+1}^{\infty} \xi_i^2 \leq \sum_{i=m+1}^{\infty} \xi_i^2 \left(\frac{i}{m+1}\right)^{2q} \\
&\leq \frac{1}{\pi^{2q}\left(m+1\right)^{2q}}\pi^{2q}\sum_{i=1}^{\infty} \xi_i^2 i^{2q} = \frac{\int_0^1 \left(\varphi^{(q)}(x)\right)^2 dx}{\pi^{2q}\left(m+1\right)^{2q}}.
\end{aligned}$$

1.2 The Stochastic Integral $\int_0^1 \cos(\ell\pi x) W(x) dW'(x)$

The matrix $\int_0^1 \cos(\ell\pi x) W(x) dW'(x)$, $\ell = 1, 2, ...$, is a $k \times k$ matrix of random variables whose (i, j)-th element is the scalar integral $\int_0^1 \cos(\ell\pi x) W_i(x) dW_j(x)$. We first claim that for an arbitrary entry (i, j),

$$C(x, \omega) = \cos(\ell\pi x) W(x, \omega) \in \mathcal{V}^{k\times 1}(0, s) \text{ for } s = 1,$$

where $\mathcal{V}^{k\times 1}(0, s)$ is the class of integrand functions V for which the Ito integral $\int_0^s V dW'$ is defined: $(x, \omega) \to C(x, \omega)$ is $\mathcal{B} \times \mathcal{F}$ measurable, $C(x, \omega)$ is $\mathcal{F}_x$ adapted, and

$$\begin{aligned} E\left[\int_0^s C(x)^2 dx\right] &= \frac{1}{4}s^2 + \frac{\sin(2\ell\pi s)}{4\ell\pi}s + \frac{[\cos(2\ell\pi s) - 1]}{2(2\ell\pi)^2} < \infty \\ &\quad \left(= \frac{1}{4}, \text{ if } s = 1, \ell \geq 1\right). \end{aligned}$$

Therefore, because $C \in \mathcal{V}(0, S)$, the Ito stochastic integral of C from 0 to s is defined as

$$I[C](\omega) = \int_0^s C(x, \omega) dW(x, \omega) = \lim_{n\to\infty} \int_0^s \phi_n(x, \omega) dW(x, \omega),$$

with limit in $L^2(P)$, where $\{\phi_n\}$ is a sequence of simple functions such that

$$\lim_{n\to\infty} E\left[\int_0^s (C(x) - \phi_n(x))^2 dx\right] = 0.$$

This condition is satisfied by taking

$$\phi_n(x, \omega) = \sum_{j=0}^{n} \cos(\ell\pi s_j) W(s_j, \omega).\mathbf{1}(s_j \leq x < s_{j+1}) \tag{7}$$

and $0 = s_0 \leq s_1 \leq ... \leq s_{n-1} \leq s_n = s$. For the chosen $\{\phi_n\}$ in (7),

$$I[C](\omega) = \lim_{s_{j+1}-s_j\to 0} \sum_{j=0}^{n} \cos(\ell\pi s_j) W(s_j, \omega) (W(s_{j+1}, \omega) - W(s_j, \omega)).$$

Moreover, by the one-dimensional Ito formula (see Oksendal 2003, page 44)[2] we have

$$I[C](\omega) = \frac{\cos(\ell\pi s)}{2} W^2(s) - \frac{\sin(\ell\pi s)}{2\ell\pi} + \frac{\ell\pi}{2} \int_0^s \sin(\ell\pi x) W^2(x) dx$$

[2] In Oksendal's (2003) notation, $X_t \equiv B_t$; $g(t, x) = \cos(\ell\pi t)x^2/2$; $Y_t = \cos(\ell\pi t)B_t^2/2$. The result follows after some manipulations.

where $\int_0^s \sin(\ell\pi x) W^2(x)dx$ is a random variable with expectation

$$E\left(\int_0^s \sin(\ell\pi x) W^2(x)dx\right) = \frac{\sin(\ell\pi s)}{(\ell\pi)^2} - \frac{\cos(\ell\pi s)}{\ell\pi}s.$$

Therefore,

$$E\left(\int_0^1 \cos(\ell\pi x) W(x,\omega)dW(x,\omega)\right) = 0.$$

The *quadratic variation process* of

$$C(t) = \int_0^t \cos(\ell\pi x) W(x) dW'(x),$$

a $k \times k$ matrix-valued martingale in continuous time with respect to $\mathcal{F}_t^{(k)}$, is now

$$\begin{aligned}\langle C\rangle(t) &= \int_0^t \mathrm{Var}\left[\cos(\ell\pi x) W(x) d'W(x) \middle| \mathcal{F}_x\right] \\ &= \int_0^t \cos^2(\ell\pi x) W(x) W(x)' dx \otimes I_k.\end{aligned}$$

2 Appendix B: Time Varying Cointegration Without Drift

2.1 Proof of Lemma 2

Recall that by the Beveridge-Nelson decomposition we can write

$$Y_t = Y_0 - V_0 + C(1)\sum_{j=1}^t U_j + V_t$$

Assumption 3 implies that $Y_0 - V_0 = 0$, but there is no need to impose this restriction here. Now

$$\frac{1}{T}\sum_{t=1}^T P_{j,T}(t) U_t Y_{t-1}' = \sqrt{2}\frac{1}{T}\sum_{t=1}^T \cos(j.\pi(t-0.5)/T) U_t' Y_{t-1} \tag{8}$$

$$= \sqrt{2}\frac{1}{T}\sum_{t=1}^T \cos(j.\pi(t-0.5)/T) U_t \sum_{j=1}^{t-1} U_j' C(1)'$$

$$+\sqrt{2}\frac{1}{T}\sum_{t=1}^{T}\cos\left(j.\pi\left(t-0.5\right)/T\right)U_{t}V_{t-1}'$$

$$+\sqrt{2}\frac{1}{T}\sum_{t=1}^{T}\cos\left(j.\pi\left(t-0.5\right)/T\right)U_{t}\left(Y_{0}-V_{0}\right)'$$

It is easy to verify that the last two terms are of order $O_p\left(1/\sqrt{T}\right)$. Moreover, it follows from Lemma A.2 that

$$\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{T}\cos\left(j.\pi\left(t/T-0.5/T\right)\right)U_{t}\sum_{j=1}^{t-1}U_{j}' \\
&= \cos\left(j.\pi\left(1-0.5/T\right)\right)\frac{1}{T}\sum_{t=1}^{T}U_{t}\sum_{j=1}^{t-1}U_{j}' \\
&+ j.\pi\int_{0}^{1}\sin\left(j.\pi\left(x-0.5/T\right)\right)\left(\frac{1}{T}\sum_{t=1}^{[xT]}U_{t}\sum_{j=1}^{t-1}U_{j}'\right)dx
\end{aligned}$$

which by Lemma A.1 and the continuous mapping theorem converges in distribution to

$$\begin{aligned}
&\cos\left[j.\pi\right]\int_{0}^{1}\left(dW\right)W' + j.\pi\int_{0}^{1}\sin\left(j.\pi x\right)\left(\int_{0}^{x}\left(dW\right)W'\right)dx \\
&= \cos\left[j.\pi\right]\int_{0}^{1}\left(dW\right)W' - \int_{0}^{1}\frac{d\cos\left(j.\pi x\right)}{dx}\left(\int_{0}^{x}\left(dW\right)W'\right)dx \\
&= \int_{0}^{1}\left(dW(x)\right)\cos\left(j.\pi x\right)W(x)'
\end{aligned}$$

The latter follows via integration by parts. The first result in Lemma 2 now follows easily from these results.

To prove the second result, observe that

$$\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{T}\cos\left(j.\pi\left(t-0.5\right)/T\right)\left(\triangle Y_{t-\ell}\right)Y_{t-1}' \\
&= \frac{1}{T}\sum_{t=1}^{T}\cos\left(j.\pi\left(t-0.5\right)/T\right)\left(\triangle Y_{t-\ell}\right)\left(Y_{t-1}-Y_{t-1-\ell}\right)' \\
&+ \frac{1}{T}\sum_{t=1}^{T}\cos\left(j.\pi\left(t-0.5\right)/T\right)\left(\triangle Y_{t-\ell}\right)Y_{t-1-\ell}'
\end{aligned}$$

Again, it follows from Lemmas A.1-A.2 that

$$\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{T}\cos\left(j.\pi\left(t-0.5\right)/T\right)\left(\triangle Y_{t-\ell}\right)Y_{t-\ell-1} \\
&\quad=\cos\left(j.\pi\left(1-0.5/T\right)\right)\frac{1}{T}\sum_{t=1}^{T}\left(\triangle Y_{t-\ell}\right)Y_{t-\ell-1} \\
&\quad+j.\pi\int_{0}^{1}\sin\left(j.\pi\left(x-0.5/T\right)\right)\left(\frac{1}{T}\sum_{t=1}^{[xT]}\left(\triangle Y_{t-\ell}\right)Y_{t-1}\right)dx \\
&\quad\overset{d}{\to}\cos\left(j.\pi\right)\left(C(1)\left(\int_{0}^{1}\left(dW\right)W'\right)C(1)'+M_0\right) \\
&\quad+j.\pi\int_{0}^{1}\sin\left(j.\pi x\right)\left(C(1)\left(\int_{0}^{x}\left(dW\right)W'\right)C(1)'+xM_0\right)dx \\
&\quad=C(1)\left(\int_{0}^{1}\left(dW\right)\cos\left(j.\pi.x\right)W'\right)C(1)'+\int_{0}^{1}\cos\left(j.\pi.x\right)dxM_0 \\
&\quad=C(1)\left(\int_{0}^{1}\left(dW\right)\cos\left(j.\pi.x\right)W'\right)C(1)'
\end{aligned}$$

Moreover, by stationarity,

$$\frac{1}{T}\sum_{t=1}^{T}\cos\left(j.\pi\left(t-0.5\right)/T\right)\left(\triangle Y_{t-\ell}\right)\left(Y_{t-1}-Y_{t-1-\ell}\right)'$$

converges in probability to a matrix $M_{j,\ell}$. The second result of Lemma 2 follows now easily.

Finally, it follows from Lemma A.2 that

$$\begin{aligned}
&\frac{1}{T^2}\sum_{t=1}^{T}P_{i,T}\left(t\right)P_{j,T}\left(t\right)Y_{t-1}Y_{t-1}' \\
&\quad=2\frac{1}{T^2}\sum_{t=1}^{T}\cos\left(i.\pi\left(t-0.5\right)/T\right)\cos\left(j.\pi\left(t-0.5\right)/T\right)Y_{t-1}Y_{t-1}' \\
&\quad=2\cos\left(i.\pi\left(1-0.5/T\right)\right)\cos\left(j.\pi\left(1-0.5/T\right)\right)\frac{1}{T^2}\sum_{t=1}^{T}Y_{t-1}Y_{t-1}' \\
&\quad-2\int_{0}^{1}\frac{d}{dx}\left(\cos\left(i.\pi\left(x-0.5/T\right)\right)\cos\left(j.\pi\left(x-0.5/T\right)\right)\right)
\end{aligned}$$

$$\times \left(\frac{1}{T^2} \sum_{t=1}^{[xT]} Y_{t-1} Y_{t-1}' \right) dx \tag{9}$$

As is well known, under Assumption 1,

$$\frac{1}{T^2} \sum_{t=1}^{[xT]} Y_{t-1} Y_{t-1}' \Rightarrow C(1) \int_0^x W(y) W(y)' dy C(1)'$$

hence by the continuous mapping theorem,

$$\begin{aligned}
&\frac{1}{T^2} \sum_{t=1}^{T} P_{i,T}(t) Y_{t-1} P_{j,T}(t) Y_{t-1}' \\
&\xrightarrow{d} 2\cos(i.\pi)\cos(j.\pi) C(1) \int_0^1 W(x) W(x)' dx C(1)' \\
&- 2C(1) \int_0^1 \frac{d}{dx} \left(\cos(i.\pi x)\cos(j.\pi x)\right) \int_0^x W(y) W(y)' dy C(1)' \\
&= 2C(1) \int_0^1 \cos(i.\pi x) W(x) \cos(j.\pi x) W(x)' dx C(1)'
\end{aligned}$$

where again the equality follows via integration by parts. This completes the proof of Lemma 2.

2.2 Proof of Lemma A.3

The existence of the probability limits $\Sigma_{\beta\beta}$, $\Sigma_{X\beta}$ and Σ_{XX} follows straightforwardly from Assumptions 1-2, and the nonsingularity of Σ_{XX} follows straightforwardly from Assumption 5. Note that $\Sigma_{\beta\beta}^*$ is the variance matrix of the residual ς_t of the linear projection of $\beta' Y_{t-1}$ on X_t: $\beta' Y_{t-1} = \Pi_0 X_t + \varsigma_t$, say. Therefore, if this variance matrix were singular then there exists a vector δ such that $\delta'\beta' Y_{t-1} = \delta'\Pi_0 X_t$ a.s. Assumption 5 excludes this.

As to the probability limit $\Sigma_{\beta\otimes I_{m+1},\beta\otimes I_{m+1}}$, observe that similar to (9),

$$\begin{aligned}
&\frac{1}{T} \sum_{t=1}^{T} P_{i,T}(t) P_{j,T}(t) \beta' Y_{t-1} Y_{t-1}' \beta \\
&= 2\frac{1}{T} \sum_{t=1}^{T} \cos(i.\pi(t-0.5)/T) \cos(j.\pi(t-0.5)/T) \beta' Y_{t-1} Y_{t-1}' \beta
\end{aligned}$$

$$= 2\cos\left(i.\pi\left(1-0.5/T\right)\right)\cos\left(j.\pi\left(1-0.5/T\right)\right)\frac{1}{T}\sum_{t=1}^{T}\beta'Y_{t-1}Y_{t-1}'\beta$$
$$-2\int_0^1 \frac{d}{dx}\left(\cos\left(i.\pi\left(x-0.5/T\right)\right)\cos\left(j.\pi\left(x-0.5/T\right)\right)\right)$$
$$\times\left(\frac{1}{T}\sum_{t=1}^{[xT]}\beta'Y_{t-1}Y_{t-1}'\beta\right)dx$$

for $i,j \geq 1$. We have already established that

$$\frac{1}{T}\sum_{t=1}^{T}\beta'Y_{t-1}Y_{t-1}'\beta = \Sigma_{\beta\beta} + o_p(1).$$

Moreover, it is not hard to verify that

$$p\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{[xT]}\beta'Y_{t-1}Y_{t-1}'\beta = x.\Sigma_{\beta\beta}$$

pointwise in $x \in [0,1]$. It follows therefore by bounded convergence and integration by parts that

$$\begin{aligned}
&p\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}P_{i,T}\left(t\right)P_{j,T}\left(t\right)\beta'Y_{t-1}Y_{t-1}'\beta \\
&= 2\left(\cos\left(i.\pi\right)\cos\left(j.\pi\right)-\int_0^1 x\frac{d}{dx}\left(\cos\left(i.\pi x\right)\cos\left(j.\pi x\right)\right)dx\right)\Sigma_{\beta\beta} \\
&= 2\int_0^1\left(\cos\left(i.\pi x\right)\cos\left(j.\pi x\right)\right)dx.\Sigma_{\beta\beta} \\
&= \left(\int_0^1\cos\left(\left(i+j\right)\pi x\right)dx+\int_0^1\cos\left(\left(i-j\right)\pi x\right)dx\right)\Sigma_{\beta\beta} \\
&= \left(\frac{\sin\left(\left(i+j\right)\pi\right)}{\left(i+j\right)\pi}+\frac{\sin\left(\left(i-j\right)\pi\right)}{\left(i-j\right)\pi}\right)\Sigma_{\beta\beta} \\
&= \begin{cases}\Sigma_{\beta\beta} & \text{if} \quad i=j,\\ O_{r,r} & \text{if} \quad i\neq j.\end{cases}
\end{aligned}$$

Similarly, for $i = 0$ and $j \geq 1$,

$$p\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}P_{j,T}\left(t\right)P_{0,T}\left(t\right)\beta'Y_{t-1}Y_{t-1}'\beta \tag{10}$$

$$= \sqrt{2}\left(\cos(j.\pi) - \int_0^1 x\frac{d}{dx}\left(\cos(j.\pi x)\right)dx\right)\Sigma_{\beta\beta}$$
$$= \sqrt{2}\int_0^1 \cos(j.\pi x)\,dx.\Sigma_{\beta\beta} = \sqrt{2}\frac{\sin(j\pi)}{j\pi}\Sigma_{\beta\beta} = O_{r,r.m}$$

Hence, $\Sigma_{\beta\otimes I_{m+1},\beta\otimes I_{m+1}} = \Sigma_{\beta\beta}\otimes I_{m+1}$. Moreover, note that $\Sigma_{\beta,\beta\otimes I_{m+1}}$ is the matrix formed by the first r rows of $\Sigma_{\beta\otimes I_{m+1},\beta\otimes I_{m+1}}$. Thus, $\Sigma_{\beta,\beta\otimes I_{m+1}} = (\Sigma_{\beta\beta}, O_{r,r.m})$. The result for $\Sigma_{X,\beta\otimes I_{m+1}}$ follows by replacing $P_{0,T}(t)\beta' Y_{t-1}$ in (10) by X_t.

Finally, since $\Sigma^*_{\beta\beta}$ is nonsingular, so is $\Sigma_{\beta\beta}$, and therefore $\Sigma^*_{\beta\otimes I_{m+1},\beta\otimes I_{m+1}}$ is nonsingular.

2.3 Proof of Lemma A.7

Substituting $\triangle Y_t = \alpha\beta' Y_{t-1} + \Gamma X_t + C_0 U_t$ in the expression for $S^{(m)}_{01,T}$ yields

$$\begin{aligned} S^{(m)}_{01,T} &= \alpha\frac{1}{T}\sum_{t=1}^{T}\beta' Y_{t-1}Y^{(m)\prime}_{t-1} \\ &\quad -\alpha\left(\frac{1}{T}\sum_{t=1}^{T}\beta' Y_{t-1}X_t'\right)\left(\frac{1}{T}\sum_{t=1}^{T}X_tX_t'\right)^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}X_tY^{(m)\prime}_{t-1}\right) \\ &\quad +C_0\frac{1}{T}\sum_{t=1}^{T}U_tY^{(m)\prime}_{t-1} \\ &\quad -C_0\left(\frac{1}{T}\sum_{t=1}^{T}U_tX_t'\right)\left(\frac{1}{T}\sum_{t=1}^{T}X_tX_t'\right)^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}X_tY^{(m)\prime}_{t-1}\right) \end{aligned} \tag{11}$$

hence

$$\begin{aligned} \left(\alpha_\perp'\Omega\alpha_\perp\right)^{-1/2}\alpha_\perp' S^{(m)}_{01,T} &= \left(\alpha_\perp'\Omega\alpha_\perp\right)^{-1/2}\alpha_\perp' C_0\frac{1}{T}\sum_{t=1}^{T}U_tY^{(m)\prime}_{t-1} \\ &\quad -\left(\alpha_\perp'\Omega\alpha_\perp\right)^{-1/2}\alpha_\perp' C_0\left(\frac{1}{T}\sum_{t=1}^{T}U_tX_t'\right)\left(\frac{1}{T}\sum_{t=1}^{T}X_tX_t'\right)^{-1} \\ &\quad \times\left(\frac{1}{T}\sum_{t=1}^{T}X_tY^{(m)\prime}_{t-1}\right). \end{aligned} \tag{12}$$

It follows now straightforwardly from (12) and Lemma A.1 that

$$\left(\alpha_\perp' \Omega \alpha_\perp\right)^{-1/2} \alpha_\perp' S_{01,T}^{(m)} = \alpha_\perp' C_0 \frac{1}{T} \sum_{t=1}^{T} U_t Y_{t-1}^{(m)\prime} + o_p(1)$$
$$\xrightarrow{d} \left(\alpha_\perp' \Omega \alpha_\perp\right)^{-1/2} \alpha_\perp' C_0 \int_0^1 (dW)\, \widetilde{W}_m' \left(C(1) \otimes I_{m+1}\right),$$

hence

$$\begin{aligned}
&\left(\alpha_\perp' \Omega \alpha_\perp\right)^{-1/2} \alpha_\perp' S_{01,T}^{(m)} \left(\beta_\perp' \otimes I_{m+1}\right) \\
&\xrightarrow{d} \left(\alpha_\perp' \Omega \alpha_\perp\right)^{-1/2} \alpha_\perp' C_0 \int_0^1 (dW)\, \widetilde{W}_m' \left(\left(C_0' \alpha_\perp \left(\alpha_\perp' \Omega \alpha_\perp\right)^{-1/2}\right) \otimes I_{m+1}\right) \\
&= \int_0^1 (dW_{k-r})\, \widetilde{W}_{k-r,m}'
\end{aligned}$$

Next, it follows from (12),

$$\begin{aligned}
\frac{1}{\sqrt{T}} \sum_{t=1}^{T} U_t Y_{t-1}^{(m)\prime} \left(\beta \otimes I_{m+1}\right) &= O_p(1), \\
\frac{1}{\sqrt{T}} \sum_{t=1}^{T} U_t X_t' &= O_p(1)
\end{aligned}$$

and Lemma A.3 that

$$\begin{aligned}
&\sqrt{T} \left(\alpha_\perp' \Omega \alpha_\perp\right)^{-1/2} \alpha_\perp' S_{01,T}^{(m)} \left(\beta \otimes I_{m+1}\right) \qquad (13) \\
&= \left(\alpha_\perp' \Omega \alpha_\perp\right)^{-1/2} \alpha_\perp' C_0 \frac{1}{\sqrt{T}} \sum_{t=1}^{T} U_t Y_{t-1}^{(m)\prime} \left(\beta \otimes I_{m+1}\right) \\
&- \left(\alpha_\perp' \Omega \alpha_\perp\right)^{-1/2} \alpha_\perp' C_0 \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} U_t X_t'\right) \left(\frac{1}{T} \sum_{t=1}^{T} X_t X_t'\right)^{-1} \\
&\quad \times \left(\frac{1}{T} \sum_{t=1}^{T} X_t Y_{t-1}^{(m)\prime} \left(\beta \otimes I_{m+1}\right)\right) \\
&= \left(\alpha_\perp' \Omega \alpha_\perp\right)^{-1/2} \alpha_\perp' C_0 \frac{1}{\sqrt{T}} \sum_{t=1}^{T} U_t Y_{t-1}^{(m)\prime} \left(\beta \otimes I_{m+1}\right)
\end{aligned}$$

$$- \left(\alpha_{\perp}'\Omega\alpha_{\perp}\right)^{-1/2}\alpha_{\perp}'C_0\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}U_tX_t'\right)\left(\Sigma_{XX}^{-1}\Sigma_{X\beta},O_{k(p-1),r.m}\right)$$
$$+ o_p(1)$$
$$= \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\vartheta_t\left(Y_{t-1}^{(m)\prime}\left(\beta\otimes I_{m+1}\right)-\left(X_t'\Sigma_{XX}^{-1}\Sigma_{X\beta},0_{r.m}'\right)\right)+o_p(1)$$

where $\vartheta_t = \left(\alpha_{\perp}'\Omega\alpha_{\perp}\right)^{-1/2}\alpha_{\perp}'C_0U_t \sim$ i.i.d. $N_{k-r}\left[0, I_{k-r}\right]$. The result

$$\sqrt{T}\left(\alpha_{\perp}'\Omega\alpha_{\perp}\right)^{-1/2}\alpha_{\perp}'S_{01,T}^{(m)}\left(\beta\otimes I_{m+1}\right)\xrightarrow{d}Z$$

follows now from McLeish's (1974) martingale difference central limit theorem.

Let $\vartheta_{i,t}$ be component i of ϑ_t. Then

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\vartheta_{it}\left(\left(\beta'\otimes I_{m+1}\right)Y_{t-1}^{(m)}-\left(\begin{array}{c}\Sigma_{X\beta}'\Sigma_{XX}^{-1}X_t\\ O_{r.m,1}\end{array}\right)\right)$$

converges in distribution to column i of Z'. The normality of Z then follows from McLeish's (1974) central limit theorem and Lemma A.3, and the independence of the columns of Z' follows from the independence of the components of ϑ_t.

Finally, it follows from (55) and Lemmas A.4-A.3 that

$$\left(\alpha'\Omega^{-1}\alpha\right)^{-1}\alpha'\Omega^{-1}S_{01,T}^{(m)} = \frac{1}{T}\sum_{t=1}^{T}\beta'Y_{t-1}Y_{t-1}^{(m)\prime} \tag{14}$$
$$-\left(\frac{1}{T}\sum_{t=1}^{T}\beta'Y_{t-1}X_t'\right)\left(\frac{1}{T}\sum_{t=1}^{T}X_tX_t'\right)^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}X_tY_{t-1}^{(m)\prime}\right)$$
$$+\left(\alpha'\Omega^{-1}\alpha\right)^{-1}\alpha'\Omega^{-1}C_0\frac{1}{T}\sum_{t=1}^{T}U_tY_{t-1}^{(m)\prime}$$
$$-\left(\alpha'\Omega^{-1}\alpha\right)^{-1}\alpha'\Omega^{-1}C_0\left(\frac{1}{T}\sum_{t=1}^{T}U_tX_t'\right)\left(\frac{1}{T}\sum_{t=1}^{T}X_tX_t'\right)^{-1}$$
$$\times\left(\frac{1}{T}\sum_{t=1}^{T}X_tY_{t-1}^{(m)\prime}\right)$$

$$= \frac{1}{T}\sum_{t=1}^{T}\beta' Y_{t-1}Y_{t-1}^{(m)\prime} - \Sigma_{X\beta}'\Sigma_{XX}^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}X_t Y_{t-1}^{(m)\prime}\right)$$
$$+ \left(\alpha'\Omega^{-1}\alpha\right)^{-1}\alpha'\Omega^{-1}C_0\frac{1}{T}\sum_{t=1}^{T}U_t Y_{t-1}^{(m)\prime} + o_p(1)$$

Now

$$\left(\alpha'\Omega^{-1}\alpha\right)^{-1}\alpha'\Omega^{-1}S_{01,T}^{(m)}\left(\beta_\perp'\otimes I_{m+1}\right) \xrightarrow{d} M$$

follows straightforwardly from (14) and Lemma A.4, and

$$\left(\alpha'\Omega^{-1}\alpha\right)^{-1}\alpha'\Omega^{-1}S_{01,T}^{(m)}\left(\beta\otimes I_{m+1}\right) = \left(\Sigma_{\beta\beta}, O_{r,r.m}\right) + o_p(1)$$

follows straightforwardly from (14) and Lemma A.3.

2.4 Proof of Lemma A.9

It follows from Lemmas A.1 and A.3 that

$$\begin{aligned} S_{11,T}^{(m)} &= \frac{1}{T}\sum_{t=1}^{T}Y_{t-1}^{(m)}Y_{t-1}^{(m)\prime} - \left(\frac{1}{T}\sum_{t=1}^{T}Y_{t-1}^{(m)}X_t'\right)\Sigma_{XX}^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}X_t Y_{t-1}^{(m)\prime}\right) \\ &\quad + o_p(1) \\ &= \frac{1}{T}\sum_{t=1}^{T}Y_{t-1}^{(m)}Y_{t-1}^{(m)\prime} + O_p(1) \end{aligned}$$

Moreover, it follows from Lemma A.1 and Lemma 2 that

$$\begin{aligned} &\frac{1}{T}\left(\beta_\perp'\otimes I_{m+1}\right)S_{11,T}^{(m)}\left(\beta_\perp'\otimes I_{m+1}\right) \\ &\xrightarrow{d} \left(\left(C_0'\alpha_\perp\left(\alpha_\perp'\Omega\alpha_\perp\right)^{-1/2}\right)\otimes I_{m+1}\right)'\int_0^1 \widetilde{W}_m(x)\widetilde{W}_m'(x)dx \\ &\qquad \times\left(\left(C_0'\alpha_\perp\left(\alpha_\perp'\Omega\alpha_\perp\right)^{-1/2}\right)\otimes I_{m+1}\right) \\ &= \int_0^1 \widetilde{W}_{k-r,m}(x)\widetilde{W}_{k-r,m}'(x)dx \end{aligned}$$

Furthermore, it is not hard to verify from Lemma 2 that

$$S_{11,T}^{(m)}\left(\beta\otimes I_{m+1}\right) = O_p(1)$$

hence

$$p \lim_{T\to\infty} \frac{1}{\sqrt{T}} \left(\beta'_{\perp} \otimes I_{m+1}\right) S^{(m)}_{11,T} \left(\beta \otimes I_{m+1}\right) = O_{k-r,r}$$

Finally, it follows from Lemma A.3 that

$$\begin{aligned}
& p \lim_{T\to\infty} \left(\beta' \otimes I_{m+1}\right) S^{(m)}_{11,T} \left(\beta \otimes I_{m+1}\right) \qquad (15) \\
&= p \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \left(\beta' \otimes I_{m+1}\right) Y^{(m)}_{t-1} Y^{(m)\prime}_{t-1} \left(\beta \otimes I_{m+1}\right) \\
&\quad - \left(p \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \left(\beta' \otimes I_{m+1}\right) Y^{(m)}_{t-1} X'_t\right) \Sigma^{-1}_{XX} \\
&\quad \times \left(p \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} X_t Y^{(m)\prime}_{t-1} \left(\beta \otimes I_{m+1}\right)\right) \\
&= \Sigma_{\beta\otimes I_{m+1},\beta\otimes I_{m+1}} - \Sigma_{\beta\otimes I_{m+1}X} \Sigma^{-1}_{XX} \Sigma_{X,\beta\otimes I_{m+1}} \\
&= \begin{pmatrix} \Sigma^{*}_{\beta\beta} & O_{r,r.m} \\ O_{r.m,r} & \Sigma_{\beta\beta} \otimes I_m \end{pmatrix}
\end{aligned}$$

2.5 Proof of Lemma A.11

It follows from Lemma A.7 that

$$\begin{aligned}
\widehat{V}_i &= \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \vartheta_{i,t} \left(\Sigma^{-1/2}_{\beta\beta} \otimes I_m\right) \left(O_{r.m,r}, I_{r.m}\right) \left(\beta' \otimes I_{m+1}\right) Y^{(m)}_{t-1} \\
&\xrightarrow{d} V_i \sim N_{r.m}\left[0, I_{r.m}\right],
\end{aligned}$$

where V_i is column i of V, and $\vartheta_{i,t}$ is component i of $\vartheta_t = \left(\alpha'_{\perp}\Omega\alpha_{\perp}\right)^{-1/2} \alpha'_{\perp} C_0 U_t$. Moreover, note that

$$\widehat{W}_{i,k-r}(x) = \frac{1}{\sqrt{T}} \sum_{t=1}^{[xT]} \vartheta_{it} \Rightarrow W_{i,k-r}(x)$$

where $W_{i,k-r}$ is component i of W_{k-r}. To prove that V_i and $W_{j,k-r}(x)$ are independent, consider the empirical process

$$\widetilde{V}_i(x) = \frac{1}{\sqrt{T}} \sum_{t=1}^{[xT]} \vartheta_{i,t} \left(\Sigma^{-1/2}_{\beta\beta} \otimes I_m\right) \left(O_{r.m,r}, I_{r.m}\right) \left(\beta' \otimes I_{m+1}\right) Y^{(m)}_{t-1}$$

Clearly, $\widetilde{V}_i \Rightarrow \overline{V}_i$, where $\overline{V}_i(.)$ is a $r.m$ variate standard Wiener process, and $V_i = \overline{V}_i(1)$. It suffices to show that for all $x, y \in [0,1]$ and $i, j = 1, ..., k-r$, $E\left[\widetilde{V}_i(x)\widehat{W}_{j,k-r}(y)\right] \to 0$. This is trivial for $i \neq j$. For $i = j$,

$$\begin{aligned} E\left[\widetilde{V}_i(x)\widehat{W}_{i,k-r}(y)\right] &= \left(\Sigma_{\beta\beta}^{-1/2} \otimes I_m\right) \\ &\times \frac{1}{T}\sum_{t=1}^{\min([xT],[yT])} E\left[(O_{r.m,r}, I_{r.m})\left(\beta' \otimes I_{m+1}\right) Y_{t-1}^{(m)}\right] \\ &= \left(\Sigma_{\beta\beta}^{-1/2} \otimes I_m\right) \\ &\times \frac{1}{T}\sum_{t=1}^{\min([xT],[yT])} \left(P_{1,T}(t) E\left[Y_{t-1}'\beta\right], ..., P_{m,T}(t) E\left[Y_{t-1}'\beta\right]\right)' \\ &= 0 \end{aligned}$$

because by Assumptions 1-3, $E\left[Y_{t-1}'\beta\right] = 0$. This proves the independence of V and W_{k-r}. The proof of the independence of V and $\widetilde{W}_{k-r,m}$ is similar.

2.6 Proof of Theorem 2

2.6.1 Data Generating Process

Recall that the data-generating process involved is

$$Z_t = (Z_{1,t}, Z_{2,t})'$$

where $Z_{1,t} \in \mathbb{R}$ and $Z_{2,t} \in \mathbb{R}$ are assumed to be generated by

$$\begin{aligned} \Delta Z_{1,t} &= b_1 Z_{1,t-1} + b_2(t/T) Z_{2,t-1} + U_{1,t} \\ \Delta Z_{2,t} &= U_{2,t} \\ U_t &= (U_{1,t}, U_{2,t})' \sim \text{ i.i.d. } N_2[0, I_2], \end{aligned} \tag{16}$$

Moreover, for some $m > 0$,

$$b_1^{-1} b_2(t/T) = \sum_{j=0}^{m} \rho_j P_{j,T}(t), \ \ \rho' = (\rho_0, \rho_1, ..., \rho_m)$$

where the $P_{j,T}(t)$'s are Chebyshev's time polynomials. Then

$$(b_1, b_2(t/T)) = b_1 \sum_{j=0}^{m} \varsigma_j' P_{j,T}(t),$$

where $\varsigma_0' = (1, \rho_0)$, and $\varsigma_j' = (0, \rho_j)$ for $j \geq 1$. Hence,

$$\begin{aligned} \Delta Z_{1,t} &= b_1 \left(Z_{1,t-1} + \sum_{j=0}^{m} \rho_j P_{j,T}(t) Z_{2,t-1} \right) + U_{1,t} \\ &= b_1 \sum_{j=0}^{m} \varsigma_j' P_{j,T}(t) Z_{t-1} + U_{1,t} = b_1 \varsigma' Z_{t-1}^{(m)} + U_{1,t} \\ \Delta Z_{2,t} &= U_{2,t} \end{aligned}$$

where

$$\varsigma' = (1, \rho_0, 0, \rho_1, 0, \rho_2, \ldots, 0, \rho_m) \tag{17}$$

and

$$Z_{t-1}^{(m)} = \begin{pmatrix} Z_{1,t-1}^{(m)} \\ Z_{2,t-1}^{(m)} \end{pmatrix} = Z_{t-1} \otimes \widehat{p}_m(t/T) \tag{18}$$

with

$$Z_{i,t-1}^{(m)} = \left(Z_{i,t-1}', P_{1,T}(t) Z_{i,t-1}', P_{2,T}(t) Z_{i,t-1}', ..., P_{m,T}(t) Z_{i,t-1} \right)', \ i = 1, 2,$$

and

$$\widehat{p}_m(t/T) = (1, P_{1,T}(t), ..., P_{m,T}(t))'$$

We can now write the model in VECM(1) form as

$$\Delta Z_t = \delta \varsigma' Z_{t-1}^{(m)} + U_t \tag{19}$$

where

$$\delta = \begin{pmatrix} b_1 \\ 0 \end{pmatrix} \tag{20}$$

In the sequel we will refer to this model, together with the applicable parts of Assumptions 1-2, as $H_1^{(m)}(p = 1)$.

Under $H_1^{(m)}(p = 1)$ the matrices $S_{00,T}$, $S_{11,T}^{(m)}$ and $S_{01,T}^{(m)}$ become

$$\begin{aligned} S_{00,T} &= \frac{1}{T} \sum_{t=1}^{T} \triangle Z_t \triangle Z_t' & (21) \\ S_{11,T}^{(m)} &= \frac{1}{T} \sum_{t=1}^{T} Z_{t-1}^{(m)} Z_{t-1}^{(m)\prime} & (22) \\ S_{01,T}^{(m)} &= \frac{1}{T} \sum_{t=1}^{T} \triangle Z_t Z_{t-1}^{(m)\prime} & (23) \end{aligned}$$

respectively

2.6.2 The matrix $S_{11,T}^{(0)}$

Under $H_1^{(m)}(p=1)$ the results of Lemma 6 in the paper read:

Lemma B.1. *Under* $H_1^{(m)}(p=1)$,

$$\Delta Z_{1,t} = \sum_{j=0}^{t-1} (1+b_1)^j \left(b_2\left((t-j)/T\right) - b_2(0)\right) U_{2,t-1-j} + V_t$$

and

$$\begin{aligned} & b_1 Z_{1,t-1} + b_2(t/T)\, Z_{2,t-1} \\ & \quad = \sum_{j=0}^{t-1} (1+b_1)^j \left(b_2\left((t-j)/T\right) - b_2(0)\right) U_{2,t-1-j} + V_t - U_{1,t} \end{aligned}$$

where

$$V_t = b_2(0) \sum_{j=0}^{\infty} (1+b_1)^j U_{2,t-1-j} + b_1 \sum_{j=0}^{\infty} (1+b_1)^j U_{1,t-1-j} + U_{1,t}$$

Moreover, it follows from Lemma B.1 that

$$\begin{aligned} b_1 \varsigma' Z_{t-1}^{(m)} &= b_1 \left(Z_{1,t-1} + \sum_{j=0}^{m} \rho_j P_{j,T}(t)\, Z_{2,t-1} \right) && (24) \\ &= b_1 Z_{1,t-1} + b_2(t/T)\, Z_{2,t-1} \\ &= \sum_{j=0}^{t-1} (1+b_1)^j \left(b_2\left((t-j)/T\right) - b_2(0)\right) U_{2,t-1-j} + R_t \end{aligned}$$

where

$$R_t = b_2(0) \sum_{j=0}^{\infty} (1+b_1)^j U_{2,t-1-j} + b_1 \sum_{j=0}^{\infty} (1+b_1)^j U_{1,t-1-j} \qquad (25)$$

Furthermore, we have

Lemma B.2. *Under* $H_1^{(m)}(p=1)$,

$$Z_{[xT]}/\sqrt{T} \Rightarrow \begin{pmatrix} -b_1^{-1} b_2(x) \\ 1 \end{pmatrix} W_2(x)$$

where $W_2(x)$ is a standard Wiener process, hence

$$\begin{aligned}\frac{1}{T}S^{(0)}_{11,T} &= \frac{1}{T^2}\sum_{t=1}^{T} Z_{t-1}Z'_{t-1} \\ &\xrightarrow{d} \begin{pmatrix} b_1^{-2}\int_0^1 b_2\left(x\right)^2 W_2(x)^2 dx & -b_1^{-1}\int_0^1 b_2\left(x\right) W_2(x)^2 dx \\ -b_1^{-1}\int_0^1 b_2\left(x\right) W_2(x)^2 dx & \int_0^1 W_2(x)^2 dx \end{pmatrix}\end{aligned}$$

Proof: Note that the model for $Z_{1,t}$ reads

$$\begin{aligned} Z_{1,t} &= \left(1+b_1\right) Z_{1,t-1} + b_2\left(t/T\right) Z_{2,t-1} + U_{1,t} \\ &= \sum_{j=0}^{t-1}\left(1+b_1\right)^j b_2\left((t-j)/T\right) Z_{2,t-1-j} + b_2\left(0\right)\sum_{j=t}^{\infty}\left(1+b_1\right)^j Z_{2,t-1-j} \\ &\quad + \sum_{j=0}^{\infty}\left(1+b_1\right)^j U_{1,t-j} \\ &= \sum_{j=0}^{t-1}\left(1+b_1\right)^j b_2\left((t-j)/T\right)\left(Z_{2,t-1-j} - Z_{2,0}\right) \\ &\quad + \sum_{j=0}^{\infty}\left(1+b_1\right)^j U_{1,t-j} \\ &\quad + Z_{2,0}\sum_{j=0}^{t-1}\left(1+b_1\right)^j b_2\left((t-j)/T\right) + Z_{2,0} b_2\left(0\right)\sum_{j=t}^{\infty}\left(1+b_1\right)^j \\ &\quad + b_2\left(0\right)\sum_{j=t}^{\infty}\left(1+b_1\right)^j\left(Z_{2,t-1-j} - Z_{2,0}\right) \qquad (26) \\ &= \sum_{j=0}^{t-2}\left(1+b_1\right)^j b_2\left((t-j)/T\right)\left(Z_{2,t-1-j} - Z_{2,0}\right) + O_p\left(1\right) \end{aligned}$$

where the $O_p\left(1\right)$ is uniform in $t = 1, ..., T$. Hence

$$Z_{1,t}/\sqrt{T} = \sum_{j=0}^{t-2}\left(1+b_1\right)^j b_2\left((t-j)/T\right) W_{2,T}\left((t-1-j)/T\right) + O_p\left(1/\sqrt{T}\right)$$

where for $x \in [0,1]$,

$$W_{2,T}\left(x\right) = \frac{1}{\sqrt{T}}\sum_{t=1}^{[xT]} U_{2,t} \Rightarrow W_2\left(x\right).$$

Moreover, by the mean value theorem,

$$\begin{aligned}
&\left|\sum_{j=0}^{t-2}(1+b_1)^j\, b_2\left((t-j)/T\right) W_{2,T}\left((t-1-j)/T\right)\right.\\
&\left.-b_2\left(t/T\right)\sum_{j=0}^{t-2}(1+b_1)^j\, W_{2,T}\left((t-1-j)/T\right)\right|\\
&\leq \frac{1}{T}\sum_{j=0}^{t-2}|1+b_1|^j j\left|W_{2,T}\left((t-1-j)/T\right)\right| \sup_{0\leq x\leq 1}|b_2'\left(x\right)|\\
&\leq \frac{1}{T}\sum_{j=0}^{\infty}|1+b_1|^j j \sup_{0\leq x\leq 1}\left|W_{2,T}\left(x\right)\right| \sup_{0\leq x\leq 1}|b_2'\left(x\right)|\\
&= O_p\left(1/T\right)
\end{aligned}$$

where that last result follows from the fact that $|1+b_1|<1$, $\sup_{0\leq x\leq 1}\left|W_{2,T}\left(x\right)\right| \xrightarrow{d} \sup_{0\leq x\leq 1}\left|W_2\left(x\right)\right|$ and $\sup_{0\leq x\leq 1}|b_2'\left(x\right)|<\infty$. Thus,

$$\begin{aligned}
&Z_{1,t}/\sqrt{T} = b_2\left(t/T\right)\sum_{j=0}^{t-2}(1+b_1)^j\, W_{2,T}\left((t-1-j)/T\right) + O_p\left(1/\sqrt{T}\right)\\
&= b_2\left(t/T\right) W_{2,T}\left((t-1)/T\right)\sum_{j=0}^{t-2}(1+b_1)^j\\
&-b_2\left(t/T\right)\sum_{j=1}^{t-2}(1+b_1)^j\left(W_{2,T}\left((t-1)/T\right)-W_{2,T}\left((t-1-j)/T\right)\right)\\
&+O_p\left(1/\sqrt{T}\right)
\end{aligned}$$

Next, observe that

$$\begin{aligned}
&E\left[\left(\sum_{j=1}^{t-2}(1+b_1)^j\left(W_{2,T}\left((t-1)/T\right)-W_{2,T}\left((t-1-j)/T\right)\right)\right)^2\right]\\
&=\sum_{j_1=1}^{t-2}\sum_{j_2=1}^{t-2}(1+b_1)^{j_1+j_2}\, E\left[\left(W_{2,T}\left((t-1)/T\right)-W_{2,T}\left((t-1-j_1)/T\right)\right)\right.\\
&\left.\times\left(W_{2,T}\left((t-1)/T\right)-W_{2,T}\left((t-1-j_2)/T\right)\right)\right]
\end{aligned}$$

$$= \frac{1}{T}\sum_{j_1=1}^{t-2}\sum_{j_2=1}^{t-2}(1+b_1)^{j_1+j_2}\min(j_1,j_2)$$
$$\leq \frac{1}{T}\left(\sum_{j=1}^{\infty}|1+b_1|^j j\right)^2 = O(1/T)$$

Hence

$$Z_{1,t}/\sqrt{T} = b_2(t/T)W_{2,T}((t-1)/T)\frac{1-(1+b_1)^{t-1}}{-b_1} + O_p\left(1/\sqrt{T}\right).$$

It follows now easily that for $x \in [0,1]$,

$$Z_{1,[xT]}/\sqrt{T} \Rightarrow -b_1^{-1}b_2(x)W_2(x).$$

whereas it is a standard result that

$$Z_{2,[xT]}/\sqrt{T} \Rightarrow W_2(x).$$

Q.E.D.

2.6.3 The matrix $S^{(0)}_{01,T}$

It follows from (16), (24) and (25) that

$$\frac{1}{T}\sum_{t=1}^{T}\Delta Z_{1,t}Z'_{t-1}$$
$$= \frac{1}{T}\sum_{t=1}^{T}\sum_{j=0}^{t-1}(b_2((t-j)/T)-b_2(0))(1+b_1)^j U_{2,t-1-j}Z'_{t-1}$$
$$+ b_2(0)\frac{1}{T}\sum_{t=1}^{T}\left(\sum_{j=0}^{\infty}(1+b_1)^j U_{2,t-1-j}\right)Z'_{t-1}$$
$$+ b_1\frac{1}{T}\sum_{t=1}^{T}\left(\sum_{j=0}^{\infty}(1+b_1)^j U_{1,t-1-j}\right)Z'_{t-1} + \frac{1}{T}\sum_{t=1}^{T}U_{1,t}Z'_{t-1}$$
$$= \frac{1}{T}\sum_{t=1}^{T}\sum_{j=0}^{t-1}b_2((t-j)/T)(1+b_1)^j U_{2,t-1-j}Z'_{t-1}$$

$$+ b_1 \frac{1}{T} \sum_{t=1}^{T} \sum_{j=0}^{\infty} (1+b_1)^j U_{1,t-1-j} Z'_{t-1} + \frac{1}{T} \sum_{t=1}^{T} U_{1,t} Z'_{t-1}$$

$$+ b_2(0) \left(\sum_{j=0}^{\infty} (1+b_1)^j U_{2,-1-j} \right) \frac{1}{T} \sum_{t=1}^{T} (1+b_1)^t Z'_{t-1}$$

Using Lemma A.2 in the paper, we can write

$$\sum_{j=0}^{t-1} b_2((t-j)/T)(1+b_1)^j U_{2,t-1-j} Z'_{t-1}$$

$$= b_2(t/T) U_{2,t-1} Z'_{t-1}$$

$$+ \sum_{j=1}^{t-1} b_2 \left(\frac{t}{T} - \frac{j}{t-1} \times \frac{t-1}{T} \right) (1+b_1)^j U_{2,t-1-j} Z'_{t-1}$$

$$= b_2(t/T) U_{2,t-1} Z'_{t-1} + b_2(1/T) \sum_{j=1}^{t-1} (1+b_1)^j U_{2,t-1-j} Z'_{t-1}$$

$$+ \frac{t-1}{T} \int_0^1 b'_2 \left(\frac{t}{T} - x . \frac{t-1}{T} \right) \sum_{j=0}^{[x(t-1)]} (1+b_1)^j U_{2,t-1-j} Z'_{t-1} dx$$

$$= b_2(t/T) U_{2,t-1} Z'_{t-1} + b_2(1/T) \sum_{j=1}^{t-1} (1+b_1)^j U_{2,t-1-j} Z'_{t-1}$$

$$+ \frac{t-1}{T} \int_0^1 b'_2 \left(\frac{t}{T} - x . \frac{t-1}{T} \right) dx \sum_{j=0}^{\infty} (1+b_1)^j U_{2,t-1-j} Z'_{t-1}$$

$$- \frac{t-1}{T} \int_0^1 b'_2 \left(\frac{t}{T} - x . \frac{t-1}{T} \right) \sum_{j=[x(t-1)]+1}^{\infty} (1+b_1)^j U_{2,t-1-j} Z'_{t-1} dx$$

$$= b_2(t/T) U_{2,t-1} Z'_{t-1} + b_2(1/T) \sum_{j=1}^{t-1} (1+b_1)^j U_{2,t-1-j} Z'_{t-1}$$

$$- (b_2(1/T) - b_2(t/T)) \sum_{j=0}^{\infty} (1+b_1)^j U_{2,t-1-j} Z'_{t-1}$$

$$- \frac{t-1}{T} \int_0^1 b'_2 \left(\frac{t}{T} - x . \frac{t-1}{T} \right) \sum_{j=[x(t-1)]+1}^{\infty} (1+b_1)^j U_{2,t-1-j} Z'_{t-1} dx$$

$$= (b_2(t/T) - b_2(1/T)) U_{2,t-1} Z'_{t-1}$$

$$-b_2\left(1/T\right)\left(1+b_1\right)^t\sum_{j=0}^{\infty}\left(1+b_1\right)^j U_{2,-1-j}Z'_{t-1}$$

$$+b_2\left(t/T\right)\sum_{j=0}^{\infty}\left(1+b_1\right)^j U_{2,t-1-j}Z'_{t-1}$$

$$-\frac{t-1}{T}\int_0^1 b_2'\left(\frac{t}{T}-x.\frac{t-1}{T}\right)\sum_{j=[x(t-1)]+1}^{\infty}\left(1+b_1\right)^j U_{2,t-1-j}Z'_{t-1}dx$$

Thus,denoting,

$$\widetilde{U}_{1,t} = \sum_{j=0}^{\infty}\left(1+b_1\right)^j U_{1,t-j}$$

$$\widetilde{U}_{2,t} = \sum_{j=0}^{\infty}\left(1+b_1\right)^j U_{2,t-j}$$

we can write

$$\frac{1}{T}\sum_{t=1}^{T}\Delta Z_{1,t}Z'_{t-1}$$

$$=\frac{1}{T}\sum_{t=1}^{T}b_2\left(t/T\right)\left(U_{2,t-1}-\widetilde{U}_{2,t-1}\right)Z'_{t-1}$$

$$+b_1\frac{1}{T}\sum_{t=1}^{T}\widetilde{U}_{1,t-1}Z'_{t-1}+\frac{1}{T}\sum_{t=1}^{T}U_{1,t}Z'_{t-1}-b_2\left(1/T\right)\frac{1}{T}\sum_{t=1}^{T}U_{2,t-1}Z'_{t-1}$$

$$+\left(b_2\left(0\right)-b_2\left(1/T\right)\right)\widetilde{U}_{2,-1}\frac{1}{T}\sum_{t=1}^{T}\left(1+b_1\right)^t Z'_{t-1}$$

$$-\frac{1}{T}\sum_{t=1}^{T}\frac{t-1}{T}\int_0^1 b_2'\left(\frac{t}{T}-x.\frac{t-1}{T}\right)\left(1+b_1\right)^{[x(t-1)]+1}\widetilde{U}_{2,t-1-[x(t-1)]-1}dxZ'_{t-1}$$

The last two terms are $o_p\left(1/\sqrt{T}\right)$, which can be shown as follows. First, observe that

$$\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{t-1}{T}\int_0^1 b_2'\left(\frac{t-1}{T}\left(1-x\right)+\frac{1}{T}\right)\right.$$

$$\times \left(1+b_1\right)^{[x(t-1)]+1} \widetilde{U}_{2,t-1-[x(t-1)]-1} dx.Z_{t-1}\Big\|$$
$$\leq \sup_{0\leq x\leq 1} \left|b_2'\left(x\right)\right| \times \max_{1\leq t\leq T} \left\|T^{-1/2}Z_{t-1}\right\|$$
$$\times \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \frac{t-1}{T} \int_0^1 \left|1+b_1\right|^{[x(t-1)]+1} \left|\widetilde{U}_{2,t-1-[x(t-1)]-1}\right| dx$$

and

$$E\left[\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \frac{t-1}{T} \int_0^1 \left|1+b_1\right|^{[x(t-1)]+1} \left|\widetilde{U}_{2,t-1-[x(t-1)]-1}\right| dx\right]$$
$$= E\left[\left|\widetilde{U}_{2,0}\right|\right] \int_0^1 \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \frac{t-1}{T} \left|1+b_1\right|^{[x(t-1)]+1} dx$$
$$= E\left[\left|\widetilde{U}_{2,0}\right|\right] \sqrt{T} \int_0^1 \int_0^1 \frac{[yT]}{T} \left|1+b_1\right|^{[x[yT]]+1} dxdy$$
$$\leq E\left[\left|\widetilde{U}_{2,0}\right|\right] \sqrt{T} \int_0^1 \int_0^1 y \left|1+b_1\right|^{x(yT-1)} dxdy$$
$$= E\left[\left|\widetilde{U}_{2,0}\right|\right] \sqrt{T} \int_0^1 \left(\int_0^1 y \exp\left(x\left(yT-1\right)\ln\left|1+b_1\right|\right) dy\right) dx$$
$$= E\left[\left|\widetilde{U}_{2,0}\right|\right] \frac{1}{\sqrt{T}} \int_0^1 \exp\left(-x\ln\left|1+b_1\right|\right) \left(\int_0^1 yT\exp\left(xyT\ln\left|1+b_1\right|\right) dy\right) dx$$
$$= o_p\left(1/\sqrt{T}\right)$$

The latter follows from the fact that $\ln|1+b_1| < 0$ so that by bounded convergence

$$\lim_{T\to\infty} \int_0^1 yT\exp\left(xyT\ln\left|1+b_1\right|\right) dy = 0$$

pointwise in $x \in (0,1]$.

Next, observe that $b_2(0) - b_2(1/T) = o(1)$ and

$$\begin{aligned}
\left\|\frac{1}{T}\sum_{t=1}^{T}\left(1+b_1\right)^t Z_{t-1}\right\| &\leq \max_{1\leq t\leq T}\left\|T^{-1/2}Z_{t-1}\right\| \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left|1+b_1\right|^t \\
&\leq \frac{1}{\sqrt{T}} \max_{1\leq t\leq T}\left\|T^{-1/2}Z_{t-1}\right\| \frac{1}{1-\left|1+b_1\right|} \\
&= O_p\left(1/\sqrt{T}\right)
\end{aligned}$$

because it follows from Lemma B.2 that $\max_{1\leq t\leq T}\left\|T^{-1/2}Z_{t-1}\right\| = O_p(1)$. Hence

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\Delta Z_{1,t}Z_{t-1}' &= \frac{1}{T}\sum_{t=1}^{T}b_2(t/T)\left(U_{2,t-1}-\widetilde{U}_{2,t-1}\right)Z_{t-1}' \\ &+b_1\frac{1}{T}\sum_{t=1}^{T}\widetilde{U}_{1,t-1}Z_{t-1}'+\frac{1}{T}\sum_{t=1}^{T}U_{1,t}Z_{t-1}' \\ &-b_2(1/T)\frac{1}{T}\sum_{t=1}^{T}U_{2,t-1}Z_{t-1}'+o_p\left(1/\sqrt{T}\right)\end{aligned}$$

Again, using Lemma A.2 in the paper, we can write

$$\begin{aligned}&\frac{1}{T}\sum_{t=1}^{T}b_2(t/T)\left(U_{2,t-1}-\widetilde{U}_{2,t-1}\right)Z_{t-1}' \\ &= b_2(1)\frac{1}{T}\sum_{t=1}^{T}\left(U_{2,t-1}-\widetilde{U}_{2,t-1}\right)Z_{t-1}'-\int_0^1 b_2'(x)\frac{1}{T}\sum_{t=1}^{[xT]}\left(U_{2,t-1}-\widetilde{U}_{2,t-1}\right)Z_{t-1}'\end{aligned}$$

hence

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\Delta Z_{1,t}Z_{t-1}' &= b_2(1)\frac{1}{T}\sum_{t=1}^{T}\left(U_{2,t-1}-\widetilde{U}_{2,t-1}\right)Z_{t-1}' \\ &-\int_0^1 b_2'(x)\frac{1}{T}\sum_{t=1}^{[xT]}\left(U_{2,t-1}-\widetilde{U}_{2,t-1}\right)Z_{t-1}' \\ &+b_1\frac{1}{T}\sum_{t=1}^{T}\widetilde{U}_{1,t-1}Z_{t-1}'+\frac{1}{T}\sum_{t=1}^{T}U_{1,t}Z_{t-1}' \\ &-b_2(1/T)\frac{1}{T}\sum_{t=1}^{T}U_{2,t-1}Z_{t-1}'+o_p\left(1/\sqrt{T}\right)\end{aligned}$$

It is now a standard exercise to prove that $\frac{1}{T}\sum_{t=1}^{T}\Delta Z_{1,t}Z_{2,t-1}$ converges in distribution. In particular, there exist constants σ_i,κ_i such that

$$\frac{1}{T}\sum_{t=1}^{T}\left(\widetilde{U}_{2,t-1}-U_{2,t-1}\right)Z_{2,t-1}\xrightarrow{d}\sigma_1\int_0^1 W_2(x)\,dW_2(x)+\kappa_1$$

$$\int_0^1 b_2'(x)\,\frac{1}{T}\sum_{t=1}^{[xT]}\left(\widetilde{U}_{2,t-1}-U_{2,t-1}\right)Z_{2,t-1}$$
$$\xrightarrow{d}\sigma_2\int_0^1 b_2'(x)\left(\int_0^x W_2(y)dW_2(y)\right)dx+\kappa_2$$
$$\frac{1}{T}\sum_{t=1}^{T}U_{2,t-1}Z_{2,t-1}\xrightarrow{d}\int_0^1 W_2(x)\,dW_2(x)+1$$
$$\frac{1}{T}\sum_{t=1}^{T}\widetilde{U}_{1,t-1}Z_{2,t-1}\xrightarrow{d}\sigma_3\int_0^1 W_2(x)\,dW_1(x)+\kappa_3$$
$$\frac{1}{T}\sum_{t=1}^{T}U_{1,t}Z_{2,t-1}\xrightarrow{d}\int_0^1 W_2(x)\,dW_1(x)$$

where W_1 is a standard Wiener process which is independent of W_2.

To prove that also $\frac{1}{T}\sum_{t=1}^{T}\Delta Z_{1,t}Z_{1,t-1}$ converges in distribution, observe from (26) that

$$\begin{aligned}
Z_{1,t-1} &= \sum_{j=0}^{t-2}(1+b_1)^j\,b_2\left((t-1-j)/T\right)Z_{2,t-2-j} \\
&\quad +\sum_{j=0}^{\infty}(1+b_1)^j\,U_{1,t-1-j} \\
&\quad +b_2(0)\,(1+b_1)^{t-1}\left(\sum_{j=0}^{\infty}(1+b_1)^j\,Z_{2,-1-j}\right) \\
&= \sum_{j=0}^{t-2}(1+b_1)^j\,b_2\left((t-1-j)/T\right)Z_{2,t-2-j} \\
&\quad +\widetilde{U}_{1,t-1}+b_2(0)\,\widetilde{Z}_{2,-1}\,(1+b_1)^{t-1} \\
&= \sum_{j=0}^{t-1}(1+b_1)^j\,b_2\left((t-1-j)/T\right)Z_{2,t-2-j}+\widetilde{U}_{1,t-1} \\
&\quad +b_2(0)\left(\widetilde{Z}_{2,-1}-Z_{2,-1}\right)(1+b_1)^{t-1}
\end{aligned}$$

It follows from the mean value theorem that for $3 \leq t \leq T$,

$$\begin{aligned}
&\left|\sum_{j=0}^{t-1} b_2\left((t-1-j)/T\right)(1+b_1)^j Z_{2,t-2-j}\right. \\
&\quad\left.-\sum_{j=0}^{t-1} b_2\left((t-1)/T\right)(1+b_1)^j Z_{2,t-2-j}\right| \\
&\quad\leq \sup_{0\leq x\leq 1}\left|b_2'(x)\right| \frac{1}{T}\sum_{j=0}^{t-1} j\left|1+b_1\right|^j\left|Z_{2,t-2-j}\right| \\
&\quad\leq \frac{1}{\sqrt{T}}\sup_{0\leq x\leq 1}\left|b_2'(x)\right| \times \max_{1\leq t\leq T}\left|T^{-1/2}Z_{2,t}\right| \times \sum_{j=0}^{\infty} j\left|1+b_1\right|^j \\
&\quad= O_p\left(1/\sqrt{T}\right)
\end{aligned}$$

hence

$$\begin{aligned}
Z_{1,t-1} &= b_2\left((t-1)/T\right)\sum_{j=0}^{t-1}(1+b_1)^j Z_{2,t-2-j} + \widetilde{U}_{1,t-1} \\
&\quad + b_2(0)\left(\widetilde{Z}_{2,-1} - Z_{2,-1}\right)(1+b_1)^{t-1} + O_p\left(1/\sqrt{T}\right)
\end{aligned}$$

Then, for example,

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T} U_{2,t-1}Z_{1,t-1} &= \frac{1}{T}\sum_{t=1}^{T} b_2\left((t-1)/T\right)U_{2,t-1}\sum_{j=0}^{t-1}(1+b_1)^j Z_{2,t-2-j} \\
&\quad + \frac{1}{T}\sum_{t=1}^{T} U_{2,t-1}\widetilde{U}_{1,t-1} + o_p(1)
\end{aligned}$$

Denote $\widehat{Z}_{2,t-2} = Z_{2,t-2}$ for $t \geq 1$, $\widehat{Z}_{2,t-2} = 0$ for $t < 1$. Then

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T} U_{2,t-1}Z_{1,t-1} &= \sum_{j=0}^{\infty}(1+b_1)^j \frac{1}{T}\sum_{t=1}^{T} b_2\left((t-1)/T\right)U_{2,t-1}\widehat{Z}_{2,t-2-j} \\
&\quad + \frac{1}{T}\sum_{t=1}^{T} U_{2,t-1}\widetilde{U}_{1,t-1} + o_p(1)
\end{aligned}$$

Using Lemma A.2 in the paper, we can write

$$\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{T} b_2\left((t-1)/T\right) U_{2,t-1}\widehat{Z}_{2,t-2-j} \\
&= b_2\left(1-1/T\right)\frac{1}{T}\sum_{t=1}^{T} U_{2,t-1}\widehat{Z}_{2,t-2-j} \\
&-\int_0^1 b_2'\left(x-1/T\right)\frac{1}{T}\sum_{t=1}^{[xT]} U_{2,t-1}\widehat{Z}_{2,t-2-j}dx
\end{aligned}$$

which converges in distribution or in probability. Moreover,

$$\frac{1}{T}\sum_{t=1}^{T} U_{2,t-1}\widetilde{U}_{1,t-1} \xrightarrow{p} E\left[U_{2,1}\widetilde{U}_{1,1}\right]$$

Thus $\frac{1}{T}\sum_{t=1}^{T} U_{2,t-1}Z_{1,t-1}$ converges in distribution. By a similar argument it follows that $\frac{1}{T}\sum_{t=1}^{T}\Delta Z_{1,t}Z_{1,t-1}$ converges in distribution. Along the same lines it can be shown that $\frac{1}{T}\sum_{t=1}^{T}\Delta Z_{2,t}Z_{t-1} = \frac{1}{T}\sum_{t=1}^{T} U_{2,t}Z_{t-1}$ converges in distribution. Consequently,

Lemma B.3. *Under* $H_1^{(m)}(p=1)$, $S_{01,T}^{(0)} = \frac{1}{T}\sum_{t=1}^{T}\Delta Z_t Z_{t-1}' \xrightarrow{d} S_{01}^{(0)}$, *where the latter is a random matrix.*

2.6.4 The scalar $\varsigma' S_{11,T}^{(m)}\varsigma$

It follows from (24) and (25) that

$$\begin{aligned}
b_1^2 E\left[\varsigma' Z_{t-1}^{(m)} Z_{t-1}^{(m)\prime}\varsigma\right] &= \sum_{j=0}^{t-1}\left(1+b_1\right)^{2j}\left(b_2\left((t-j)/T\right)-b_2\left(0\right)\right)^2 \\
&+2\sum_{j=0}^{t-1}\left(1+b_1\right)^{2j}\left(b_2\left((t-j)/T\right)-b_2\left(0\right)\right) \\
&+\frac{b_1^2+b_2\left(0\right)^2}{1-\left(1+b_1\right)^2}
\end{aligned}$$

Since b_2 is differentiable with bounded derivative b_2', it follows by the mean value theorem that

$$\left|\sum_{j=0}^{t-1}(1+b_1)^{2j}(b_2((t-j)/T)-b_2(0))-\sum_{j=0}^{t-1}(1+b_1)^{2j}(b_2(t/T)-b_2(0))\right|$$
$$\leq \sup_{x\in[0,1]}|b_2(x)|\sum_{j=1}^{t-1}(1+b_1)^{2j}j/T\leq T^{-1}\sup_{x\in[0,1]}|b_2(x)|\sum_{j=1}^{\infty}(1+b_1)^{2j}j$$
$$=O(1/T)$$

hence

$$\sum_{j=0}^{t-1}(1+b_1)^{2j}(b_2((t-j)/T)-b_2(0))$$
$$=\sum_{j=0}^{t-1}(1+b_1)^{2j}(b_2(t/T)-b_2(0))+O(1/T)$$
$$=\frac{1-(1+b_1)^{2t}}{1-(1+b_1)^2}(b_2(t/T)-b_2(0))+O(1/T)$$

Similarly, it follows that

$$\sum_{j=0}^{t-1}(1+b_1)^{2j}(b_2((t-j)/T)-b_2(0))^2$$
$$=\frac{1-(1+b_1)^{2t}}{1-(1+b_1)^2}(b_2(t/T)-b_2(0))^2+O(1/T)$$

Thus,

$$\lim_{T\to\infty}b_1^2E\left[\frac{1}{T}\sum_{t=1}^{T}\varsigma' Z_{t-1}^{(m)}Z_{t-1}^{(m)\prime}\varsigma\right]$$
$$=\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\frac{1-(1+b_1)^{2t}}{1-(1+b_1)^2}(b_2(t/T)-b_2(0))^2$$
$$+2\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\frac{1-(1+b_1)^{2t}}{1-(1+b_1)^2}(b_2(t/T)-b_2(0))$$
$$+\frac{b_1^2+b_2(0)^2}{1-(1+b_1)^2}$$

It follows now easily that

$$\begin{aligned} p\lim_{T\to\infty}\varsigma' S_{11,T}^{(m)}\varsigma &= p\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\varsigma' Z_{t-1}^{(m)} Z_{t-1}^{(m)\prime}\varsigma \\ &= \lim_{T\to\infty} E\left[\frac{1}{T}\sum_{t=1}^{T}\varsigma' Z_{t-1}^{(m)} Z_{t-1}^{(m)\prime}\varsigma\right] \\ &= \frac{1}{b_1^2\left(1-(1+b_1)^2\right)}\left[\int_0^1 (b_2(x)-b_2(0))^2\,dx\right. \\ &\quad \left. +2\int_0^1 (b_2(x)-b_2(0))\,dx + b_1^2 + b_2(0)^2\right] \end{aligned} \tag{27}$$

2.6.5 The matrix $S_{00,T}$

Recall that

$$\begin{aligned} S_{00,T} &= \frac{1}{T}\sum_{t=1}^{T}\triangle Z_t \triangle Z_t' \\ &= \frac{1}{T}\sum_{t=1}^{T}\left(\delta\varsigma' Z_{t-1}^{(m)} + U_t\right)\left(Z_{t-1}^{(m)\prime}\varsigma\delta' + U_t'\right) \\ &= \varsigma' S_{11,T}^{(m)}\varsigma\delta\delta' + \delta\frac{1}{T}\sum_{t=1}^{T}\varsigma' Z_{t-1}^{(m)} U_t' \\ &\quad +\frac{1}{T}\sum_{t=1}^{T} U_t Z_{t-1}^{(m)\prime}\varsigma\delta' + \frac{1}{T}\sum_{t=1}^{T} U_t U_t' \end{aligned}$$

It follows from (24) and (25) that

$$\begin{aligned} E\left[b_1\varsigma' Z_{t-1}^{(m)} U_t'\right] &= \sum_{j=0}^{t-1}(1+b_1)^j\left(b_2((t-j)/T)-b_2(0)\right)E\left[U_{2,t-1-j}U_t'\right] \\ &\quad +E\left[R_t U_t'\right] = 0 \end{aligned}$$

It therefore follows straightforwardly from (20) that

$$p\lim_{T\to\infty} S_{00,T} = \begin{pmatrix} b_1^2.p\lim_{T\to\infty}\varsigma' S_{11,T}^{(m)}\varsigma + 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{28}$$

2.6.6 The vector $S_{01,T}^{(m)}\varsigma$

The vector $S_{01,T}^{(m)}\varsigma$ is given by

$$\begin{aligned} S_{01,T}^{(m)}\varsigma &= \frac{1}{T}\sum_{t=1}^{T}\triangle Z_t Z_{t-1}^{(m)\prime}\varsigma \\ &= \frac{1}{T}\sum_{t=1}^{T}\delta\varsigma' Z_{t-1}^{(m)} Z_{t-1}^{(m)\prime}\varsigma + \frac{1}{T}\sum_{t=1}^{T} U_t Z_{t-1}^{(m)\prime}\varsigma \end{aligned}$$

Similar to the previous results it follows straightforwardly that

$$p\lim_{T\to\infty} S_{01,T}^{(m)}\varsigma = \begin{pmatrix} b_1.p\lim_{T\to\infty}\varsigma' S_{11,T}^{(m)}\varsigma \\ 0 \end{pmatrix}$$

2.6.7 Conclusions

Consequently,

$$p\lim_{T\to\infty}\widehat{\lambda}_{\max}^{(m)} = p\lim_{T\to\infty}\frac{\varsigma' S_{10,T}^{(m)} S_{00,T}^{-1} S_{01,T}^{(m)}\varsigma}{\varsigma' S_{11,T}^{(m)}\varsigma} = \frac{b_1^2.p\lim_{T\to\infty}\varsigma' S_{11,T}^{(m)}\varsigma}{b_1^2.p\lim_{T\to\infty}\varsigma' S_{11,T}^{(m)}\varsigma + 1} \in (0,1) \tag{29}$$

On the other hand, it follows from Lemmas 3-4 and (28) that

$$B_T = S_{10,T}^{(0)} S_{00,T}^{-1} S_{01,T}^{(0)} \xrightarrow{d} B$$

say, and

$$\begin{aligned} A_T &= \frac{1}{T}S_{11,T}^{(0)} \xrightarrow{d} \begin{pmatrix} b_1^{-2}\int_0^1 b_2(x)^2 W_2(x)^2 dx & -b_1^{-1}\int_0^1 b_2(x) W_2(x)^2 dx \\ -b_1^{-1}\int_0^1 b_2(x) W_2(x)^2 dx & \int_0^1 W_2(x)^2 dx \end{pmatrix} \\ &= A \end{aligned}$$

say. Hence, for any nonzero vector $\beta \in \mathbb{R}^2$,

$$T\frac{\beta' S_{10,T}^{(0)} S_{00,T}^{-1} S_{01,T}^{(0)}\beta}{\beta' S_{11,T}^{(0)}\beta} = \frac{\beta' B_T\beta}{\beta' A_T\beta} \xrightarrow{d} \frac{\beta' B\beta}{\beta' A\beta}$$

so that

$$p\lim_{T\to\infty}\frac{\beta' S_{10,T}^{(0)} S_{00,T}^{-1} S_{01,T}^{(0)}\beta}{\beta' S_{11,T}^{(0)}\beta} = 0$$

3 Appendix C: Time Varying Cointegration With Drift

3.1 Null Hypothesis

For convenience we list here the assumptions corresponding to the time-invariant cointegration with drift case.

Assumption C.1. $\triangle Y_t = C(L)(U_t + \mu) = \sum_{j=0}^{\infty} C_j (U_{t-j} + \mu)$, *with* $U_t \sim$ *i.i.d.* $N_k(0, I_k)$, *and* μ *a vector of imbedded drift parameters. Moreover, the elements of the* $k \times k$ *matrices* C_j *decrease exponentially to zero with* j.

Given Assumption C.1, we can write $\triangle Y_t$ as

$$\triangle Y_t = C(1)U_t + C(1)\mu + V_t - V_{t-1} \tag{30}$$

where

$$V_t = D(L)U_t \text{ with } D(L) = \frac{C(L) - C(1)}{1 - L} \tag{31}$$

is a zero-mean stationary Gaussian process. This is the well-known Beveridge-Nelson decomposition. Hence

$$Y_t = C(1)\sum_{j=1}^{t} U_j + C(1)\mu.t + V_t + Y_0 - V_0 \tag{32}$$

The next assumption is the same as Assumption 2 in the paper:

Assumption C.2. *The matrix* $C(1)$ *is singular, with rank* $1 \leq r < k$: *There exists a* $k \times r$ *matrix* β *such that* $\beta' C(1) = O_{r,k}$. *Moreover, the* $r \times k$ *matrix* $\beta' D(1)$ *has rank* r.

Thus under Assumptions C.1-2, $\beta' Y_t = \beta' V_t + \beta' (Y_0 - V_0)$.

Recall that Assumption 3 in the paper, stating that $U_t = 0$ for $t < 1$, has been dropped, so that now $\beta' (Y_0 - V_0) \neq 0$.

The next Assumption C.3 replaces Assumption 4 in the paper.

Assumption C.3. $\triangle Y_t$ *has the VECM*(p) *representation*

$$\triangle Y_t = \gamma_0 + \alpha\beta' Y_{t-1} + \sum_{j=1}^{p-1} \Gamma_j \triangle Y_{t-j} + C_0 U_t \tag{33}$$

Finally, Assumption 5 will be restated as Assumption C.4.

Assumption C.4. Var$\left(\left(Y'_{t-1}\beta, X'_t\right)'\right)$ *is nonsingular, where*

$$X_t = \left(\triangle Y_{t-1}, ..., \triangle Y_{t-p+1}\right)'.$$

3.2 Time Varying Error Correction Model

Let

$$\triangle Y_t = \gamma_0 + \alpha\xi' Y_{t-1}^{(m)} + \sum_{j=1}^{p-1} \Gamma_j \triangle Y_{t-j} + C_0 U_t \tag{34}$$

where $\xi' = \left(\xi_0', \xi_1', ..., \xi_m'\right)$ is an $r \times (m+1)k$ matrix of full rank, and now

$$\begin{aligned} Y_{t-1}^{(m)} &= \left(Y'_{t-1}, P_{1,T}(t) Y'_{t-1}, P_{2,T}(t) Y'_{t-1}, ..., P_{m,T}(t) Y'_{t-1}\right)' \\ &= P_T(t) \otimes Y_{t-1} \end{aligned}$$

where $P_{j,T}(t)$ is a Chebishev polynomial of order j, i.e.,

$$\begin{aligned} P_{0,T}(t) &= 1, \ P_{i,T}(t) = \sqrt{2}\cos\left(i\pi\left(t - 0.5\right)/T\right), \\ t &= 1, 2, ..., T, \ i = 1, 2, 3, ... \end{aligned} \tag{35}$$

and

$$P_T(t) = \left(1, P_{1,T}(t), P_{2,T}(t), ..., P_{m,T}(t)\right)' \tag{36}$$

Under Assumptions C.1-3,

$$\xi = \begin{pmatrix} \beta \\ O_{m.k\times r} \end{pmatrix} \tag{37}$$

where β is the $k \times r$ matrix of TI cointegrating vectors, and $Y_{t-1}^{(0)} = Y_{t-1}$, so that then (34) reduces to (33).

Denote

$$\widetilde{X}_t = \left(1, X'_t\right)', \ \Gamma = \left(\gamma_0, \Gamma_1, ..., \Gamma_{p-1}\right).$$

Then we can write (34) as

$$\triangle Y_t = \alpha\xi' Y_{t-1}^{(m)} + \Gamma\widetilde{X}_t + C_0 U_t \tag{38}$$

3.3 Properties of $Y_{t-1}^{(m)}$

Under the null hypothesis,

$$Y_{t-1}^{(m)} = P_T(t) \otimes \left(C(1) \sum_{j=1}^{t-1} U_j + C(1)\mu.\,(t-1) + V_{t-1} + Y_0 - V_0 \right)$$

where $P_T(t)$ is defined by (36).

Let $\beta_\perp$ be as in Lemma A.6, i.e., $\beta'_\perp C(1) = (\alpha'_\perp \Omega \alpha_\perp)^{-1/2} \alpha'_\perp C'_0$. Then

$$\begin{aligned}(\beta'_\perp \otimes I_{m+1})\, Y_{t-1}^{(m)} &= P_T(t) \otimes \left((\alpha'_\perp \Omega \alpha_\perp)^{-1/2} \alpha'_\perp C'_0 \sum_{j=1}^{t-1} U_j \right) \\ &\quad + P_T(t) \otimes \left((\alpha'_\perp \Omega \alpha_\perp)^{-1/2} \alpha'_\perp C'_0 \mu.\,(t-1) \right) \\ &\quad + P_T(t) \otimes (\beta'_\perp V_{t-1}) + P_T^*(t)\,(\beta'_\perp (Y_0 - V_0))\end{aligned}$$

Next, let

$$\overline{\mu} = \left(\mu' C_0 \alpha_\perp (\alpha'_\perp \Omega \alpha_\perp)^{-1} \alpha'_\perp C'_0 \mu \right)^{-1} (\alpha'_\perp \Omega \alpha_\perp)^{-1/2} \alpha'_\perp C'_0 \mu$$

which is a vector in $\mathbb{R}^{k-r}$. Note that $\overline{\mu}'\overline{\mu} = 1$ by normalization. Let $\overline{\mu}_\perp$ be an orthogonal complement of $\overline{\mu}$, normalized such that $\overline{\mu}'_\perp \overline{\mu}_\perp = I_{k-r-1}$. Then

$$\begin{aligned}&(\overline{\mu}'_\perp \otimes I_{m+1})\,(\beta'_\perp \otimes I_{m+1})\, Y_{t-1}^{(m)} \\ &= P_T(t) \otimes \left(\overline{\mu}'_\perp (\alpha'_\perp \Omega \alpha_\perp)^{-1/2} \alpha'_\perp C'_0 \sum_{j=1}^{t-1} U_j \right) \\ &+ P_T(t) \otimes (\overline{\mu}'_\perp \beta'_\perp V_{t-1}) + P_T(t)\,(\overline{\mu}'_\perp \beta'_\perp (Y_0 - V_0))\end{aligned} \tag{39}$$

and

$$\begin{aligned}&(\overline{\mu}' \otimes I_{m+1})\,(\beta'_\perp \otimes I_{m+1})\, Y_{t-1}^{(m)} \\ &= P_T(t) \otimes \left(\overline{\mu}' (\alpha'_\perp \Omega \alpha_\perp)^{-1/2} \alpha'_\perp C'_0 \sum_{j=1}^{t-1} U_j \right) \\ &+ P_T(t) \otimes (t-1) + P_T(t) \otimes (\overline{\mu}' \beta'_\perp V_{t-1}) \\ &+ P_T(t)\,(\overline{\mu}' \beta'_\perp (Y_0 - V_0))\end{aligned} \tag{40}$$

Hence

$$\left(\overline{\mu}'_{\perp}\otimes I_{m+1}\right)\left(\beta'_{\perp}\otimes I_{m+1}\right)\frac{1}{\sqrt{T}}Y^{(m)}_{[xT]}\Rightarrow p(x)\otimes\underline{W}_{k-r-1}(x) \tag{41}$$

where

$$\underline{W}_{k-r-1}=\overline{\mu}'_{\perp}\left(\alpha'_{\perp}\Omega\alpha_{\perp}\right)^{-1/2}\alpha'_{\perp}C'_0W \tag{42}$$

is a $k-r-1$ variate standard Wiener process, and

$$p(x)=\left(1,\sqrt{2}\cos(\pi x),....,\sqrt{2}\cos(m\pi x)\right)'$$

On the other hand,

$$\begin{aligned}\left(\overline{\mu}'\otimes I_{m+1}\right)\left(\beta'_{\perp}\otimes I_{m+1}\right)\frac{1}{T}Y^{(m)}_{[x.T]} &\Rightarrow p(x)\otimes\left(\overline{\mu}'\left(\alpha'_{\perp}\Omega\alpha_{\perp}\right)^{-1/2}\alpha'_{\perp}C'_0\mu\right)x\\ &= p(x)\otimes x \end{aligned} \tag{43}$$

The results (41) and (43) prove Lemma 7 and the following lemma.

Lemma C.1. *Let* $M_T=\left(T^{-1/2}\overline{\mu},\overline{\mu}_{\perp}\right)$. *Under Assumptions C.1-C.2*

$$\left(M'_T\otimes I_{m+1}\right)\left(\beta'_{\perp}\otimes I_{m+1}\right)\frac{1}{\sqrt{T}}Y^{(m)}_{[x.T]}\Rightarrow p(x)\otimes\begin{pmatrix}\underline{W}_{k-r-1}(x)\\ x\end{pmatrix}$$

hence

$$\begin{aligned}&\left(M'_T\otimes I_{m+1}\right)\left(\beta'_{\perp}\otimes I_{m+1}\right)\frac{1}{T^2}\sum_{t=1}^{T}Y^{(m)}_{t-1}Y^{(m)\prime}_{t-1}\left(\beta_{\perp}\otimes I_{m+1}\right)\left(M_T\otimes I_{m+1}\right)\\ &\xrightarrow{d}\int_0^1\left(p(x)\otimes\begin{pmatrix}\underline{W}_{k-r-1}(x)\\ x\end{pmatrix}\right)\left(p(x)\otimes\begin{pmatrix}\underline{W}_{k-r-1}(x)\\ x\end{pmatrix}\right)'dx\end{aligned}$$

This result yields the following corollary:

Lemma C.2. *Let* $\overline{Y}^{(m)}_{t-1}=Y^{(m)}_{t-1}-\frac{1}{T}\sum_{\tau=1}^{T}Y^{(m)}_{\tau-1}$. *Then under Assumptions C.1-C.4,*

$$\begin{aligned}&\left(M'_T\otimes I_{m+1}\right)\left(\beta'_{\perp}\otimes I_{m+1}\right)\frac{1}{T^2}\sum_{t=1}^{T}\overline{Y}^{(m)}_{t-1}\overline{Y}^{(m)\prime}_{t-1}\left(\beta_{\perp}\otimes I_{m+1}\right)\left(M_T\otimes I_{m+1}\right)\\ &\xrightarrow{d}\int_0^1\left(p(x)\otimes\begin{pmatrix}\underline{W}_{k-r-1}(x)\\ x\end{pmatrix}\right)\left(p(x)\otimes\begin{pmatrix}\underline{W}_{k-r-1}(x)\\ x\end{pmatrix}\right)'dx\\ &-\int_0^1\left(p(x)\otimes\begin{pmatrix}\underline{W}_{k-r-1}(x)\\ x\end{pmatrix}\right)dx\int_0^1\left(p(x)\otimes\begin{pmatrix}\underline{W}_{k-r-1}(x)\\ x\end{pmatrix}\right)'dx\end{aligned}$$

Next, observe that

$$
\begin{aligned}
\left(\beta' \otimes I_{m+1}\right) \overline{Y}_{t-1}^{(m)} &= P_T(t) \otimes \left(\beta' V_t\right) - \frac{1}{T} \sum_{\tau=1}^{T} P_T(\tau) \otimes \left(\beta' V \tau\right) \\
&= P_T(t) \otimes \left(\beta' V_t\right) + o_p(1)
\end{aligned}
$$

hence

$$
\begin{aligned}
&\left(\overline{\mu}_\perp' \otimes I_{m+1}\right)\left(\beta_\perp' \otimes I_{m+1}\right)\left(\frac{1}{T\sqrt{T}} \sum_{t=1}^{T} Y_{t-1}^{(m)} \overline{Y}_{t-1}^{(m)\prime}\right)\left(\beta \otimes I_{m+1}\right) \\
&\quad = \frac{1}{T} \sum_{t=1}^{T} \left(P_T(t) P_T(t)'\right) \otimes \left(\overline{\mu}_\perp' \left(\alpha_\perp' \Omega \alpha_\perp\right)^{-1/2} \alpha_\perp' C_0' \frac{1}{\sqrt{T}} \sum_{j=1}^{t-1} U_j\right) \\
&\quad \times \left(V_{t-1}'\beta + o_p(1)\right) \\
&\quad + \frac{1}{T\sqrt{T}} \sum_{t=1}^{T} \left(P_T(t) P_T(t)'\right) \otimes \overline{\mu}_\perp' \beta_\perp' V_{t-1} \left(V_{t-1}'\beta + o_p(1)\right) \\
&\quad + \frac{1}{T\sqrt{T}} \sum_{t=1}^{T} \left(P_T(t) P_T(t)'\right) \otimes \left(\overline{\mu}_\perp' \beta_\perp' \left(Y_0 - V_0\right)\right) \left(V_{t-1}'\beta + o_p(1)\right) \\
&\quad = o_p(1)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\left(\overline{\mu}' \otimes I_{m+1}\right)\left(\beta_\perp' \otimes I_{m+1}\right)\left(\frac{1}{T^2} \sum_{t=1}^{T} Y_{t-1}^{(m)} \overline{Y}_{t-1}^{(m)\prime}\right)\left(\beta \otimes I_{m+1}\right) \\
&\quad = \frac{1}{T} \sum_{t=1}^{T} \left(P_T(t) P_T(t)'\right) \otimes \left(\frac{t-1}{T}\right) \left(V_{t-1}'\beta + o_p(1)\right) + o_p(1) \\
&\quad = o_p(1)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\beta' \otimes I_{m+1}\right) \frac{1}{T} \sum_{t=1}^{T} \overline{Y}_{t-1}^{(m)} \overline{Y}_{t-1}^{(m)\prime} \left(\beta \otimes I_{m+1}\right) \\
&\quad = \frac{1}{T} \sum_{t=1}^{T} \left(P_T(t) P_T(t)'\right) \otimes \left(\beta' V_t V_t' \beta\right) + o_p(1)
\end{aligned}
$$

$$= \frac{1}{T}\sum_{t=1}^{T}\left(P_T(t)P_T(t)'\right)\otimes\left(\beta' E\left[V_t V_t'\right]\beta\right) + o_p(1)$$
$$= I_{m+1}\otimes\left(\beta' E\left[V_1 V_1'\right]\beta\right) + o_p(1)$$

Hence,

Lemma C.3. *Under Assumptions C.1-C.4*,

$$\left(M_T'\otimes I_{m+1}\right)\left(\beta_\perp'\otimes I_{m+1}\right)\left(\frac{1}{T\sqrt{T}}\sum_{t=1}^{T}\overline{Y}_{t-1}^{(m)}\overline{Y}_{t-1}^{(m)\prime}\right)\left(\beta\otimes I_{m+1}\right) = o_p(1),$$
$$\left(\beta'\otimes I_{m+1}\right)\frac{1}{T}\sum_{t=1}^{T}\overline{Y}_{t-1}^{(m)}\overline{Y}_{t-1}^{(m)\prime}\left(\beta\otimes I_{m+1}\right) = I_{m+1}\otimes\left(\beta' E\left[V_1 V_1'\right]\beta\right) + o_p(1)$$

3.4 The Matrices $S_{00,T}, S_{11,T}^{(m)}$ and $S_{01,T}^{(m)}$

Consider the following matrices:

$$S_{00,T} = \frac{1}{T}\sum_{t=1}^{T}\triangle Y_t \triangle Y_t' - \left(\frac{1}{T}\sum_{t=1}^{T}\triangle Y_t\widetilde{X}_t'\right)\left(\frac{1}{T}\sum_{t=1}^{T}\widetilde{X}_t\widetilde{X}_t'\right)^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}\widetilde{X}_t\triangle Y_t'\right), \tag{44}$$

$$S_{11,T}^{(m)} = \frac{1}{T}\sum_{t=1}^{T}Y_{t-1}^{(m)}Y_{t-1}^{(m)\prime} - \left(\frac{1}{T}\sum_{t=1}^{T}Y_{t-1}^{(m)}\widetilde{X}_t'\right)\left(\frac{1}{T}\sum_{t=1}^{T}\widetilde{X}_t X_t'\right)^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}\widetilde{X}_t Y_{t-1}^{(m)\prime}\right), \tag{45}$$

$$S_{01,T}^{(m)} = \frac{1}{T}\sum_{t=1}^{T}\triangle Y_t Y_{t-1}^{(m)\prime} - \left(\frac{1}{T}\sum_{t=1}^{T}\triangle Y_t\widetilde{X}_t'\right)\left(\frac{1}{T}\sum_{t=1}^{T}\widetilde{X}_t\widetilde{X}_t'\right)^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}\widetilde{X}_t\widetilde{Y}_{t-1}^{(m)\prime}\right), \tag{46}$$

$$S_{10,T}^{(m)} = \left(S_{01,T}^{(m)}\right)'. \tag{47}$$

Lemma C.4. *Let* $\triangle\overline{Y}_t = \triangle Y_t - \frac{1}{T}\sum_{\tau=1}^{T}\triangle Y\tau$ and $\overline{X}_t = X_t - \frac{1}{T}\sum_{\tau=1}^{T} X\tau$. *Then*

$$\begin{aligned} S_{00,T} &= \frac{1}{T}\sum_{t=1}^{T}\triangle\overline{Y}_t\triangle\overline{Y}_t^{'} \\ &\quad - \left(\frac{1}{T}\sum_{t=1}^{T}\triangle\overline{Y}_t\overline{X}_t^{'}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\overline{X}_t\overline{X}_t^{'}\right)^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}\overline{X}_t\triangle\overline{Y}_t^{'}\right), \quad (48) \\ S_{11,T}^{(m)} &= \frac{1}{T}\sum_{t=1}^{T}\overline{Y}_{t-1}^{(m)}\overline{Y}_{t-1}^{(m)'} \\ &\quad - \left(\frac{1}{T}\sum_{t=1}^{T}\overline{Y}_{t-1}^{(m)}\overline{X}_t^{'}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\overline{X}_t\overline{X}_t^{'}\right)^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}\overline{X}_t\overline{Y}_{t-1}^{(m)'}\right), \quad (49) \\ S_{01,T}^{(m)} &= \frac{1}{T}\sum_{t=1}^{T}\triangle\overline{Y}_t\overline{Y}_{t-1}^{(m)'} \\ &\quad - \left(\frac{1}{T}\sum_{t=1}^{T}\triangle\overline{Y}_t\overline{X}_t^{'}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\overline{X}_t\overline{X}_t^{'}\right)^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}\overline{X}_t\overline{Y}_{t-1}^{(m)'}\right), \quad (50) \end{aligned}$$

Proof: This result follows trivially from the fact the OLS residuals of a regression with intercept are the same as the OLS residuals of this regression without intercept after demeaning all the variables.

3.5 Properties of the Matrix $\frac{1}{T}\sum_{t=1}^{T}\overline{Y}_{t-1}^{(m)}\overline{X}_t'$

Now let us focus on the properties of the matrix $\frac{1}{T}\sum_{t=1}^{T}\overline{Y}_{t-1}^{(m)}\overline{X}_t'$. If follows from (39) that

$$\begin{aligned} &\left(\overline{\mu}_\perp' \otimes I_{m+1}\right)\left(\beta_\perp' \otimes I_{m+1}\right)\frac{1}{T}\sum_{t=1}^{T}\overline{Y}_{t-1}^{(m)}\overline{X}_t' \\ &\quad = \left(\overline{\mu}_\perp' \otimes I_{m+1}\right)\left(\beta_\perp' \otimes I_{m+1}\right)\frac{1}{T}\sum_{t=1}^{T}Y_{t-1}^{(m)}\overline{X}_t' \\ &\quad = \frac{1}{T}\sum_{t=1}^{T}P_T(t)\otimes\left(\overline{\mu}_\perp'\left(\alpha_\perp'\Omega\alpha_\perp\right)^{-1/2}\alpha_\perp' C_0'\sum_{j=1}^{t-1}U_j\right)\overline{X}_t' \end{aligned}$$

$$+\frac{1}{T}\sum_{t=1}^{T} P_T(t)\otimes\left(\overline{\mu}_\perp'\beta_\perp' V_{t-1}\right)\overline{X}_t'$$

$$+\frac{1}{T}\sum_{t=1}^{T} P_T(t)\left(\overline{\mu}_\perp'\beta_\perp'\left(Y_0-V_0\right)\right)\overline{X}_t'$$

The last two terms are $O_p(1)$. The first term is a matrix with typical blocks

$$\begin{aligned}
&\overline{\mu}_\perp'\left(\alpha_\perp'\Omega\alpha_\perp\right)^{-1/2}\alpha_\perp' C_0'\frac{1}{T}\sum_{t=1}^{T} P_{i,T}(t)\left(\Delta Y_{t-j}-\frac{1}{T}\sum_{\tau=1}^{T}\Delta Y_{\tau-j}\right)\left(\sum_{\tau=1}^{t-1} U\tau\right)\\
&=\overline{\mu}_\perp'\left(\alpha_\perp'\Omega\alpha_\perp\right)^{-1/2}\alpha_\perp' C_0'\frac{1}{T}\sum_{t=1}^{T} P_{i,T}(t)\left(\Delta Y_{t-j}-\frac{1}{T}\sum_{\tau=1}^{T}\Delta Y_{\tau-j}\right)\\
&\times\left(\sum_{\tau=1}^{t-j-1} U\tau\right)\\
&+\overline{\mu}_\perp'\left(\alpha_\perp'\Omega\alpha_\perp\right)^{-1/2}\alpha_\perp' C_0'\frac{1}{T}\sum_{t=1}^{T} P_{i,T}(t)\left(\Delta Y_{t-j}-\frac{1}{T}\sum_{\tau=1}^{T}\Delta Y_{\tau-j}\right)\\
&\times\left(\sum_{\tau=t-j}^{t-1} U\tau\right)\\
&=\overline{\mu}_\perp'\left(\alpha_\perp'\Omega\alpha_\perp\right)^{-1/2}\alpha_\perp' C_0'\frac{1}{T}\sum_{t=1}^{T} P_{i,T}(t)\left(\Delta Y_{t-j}-\frac{1}{T}\sum_{\tau=1}^{T}\Delta Y_{\tau-j}\right)\\
&\times\left(\sum_{\tau=1}^{t-j-1} U\tau\right)+O_p(1)
\end{aligned}$$

Using Lemma A.2 it follows that

$$\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{T} P_{i,T}(t)\left(\Delta Y_{t-j}-\frac{1}{T}\sum_{\tau=1}^{T}\Delta Y_{\tau-j}\right)\left(\sum_{\tau=1}^{t-j-1} U\tau\right)\\
&\quad=P_{i,T}(T)\frac{1}{T}\sum_{t=1}^{T}\left(\Delta Y_{t-j}-E\left[\Delta Y_1\right]\right)\left(\sum_{\tau=1}^{t-j-1} U\tau\right)\\
&\quad-\int_0^1 P_{i,T}'\left([xT]\right)\frac{1}{T}\sum_{t=1}^{[xT]}\left(\Delta Y_{t-j}-E\left[\Delta Y_1\right]\right)\left(\sum_{\tau=1}^{t-j-1} U\tau\right)dx
\end{aligned}$$

$$
\begin{aligned}
&- P_{i,T}(T)\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\Delta Y_{t-j} - E\left[\Delta Y_1\right]\right)\left(\frac{1}{T}\sum_{\tau=1}^{T}\frac{1}{\sqrt{T}}\sum_{\tau=1}^{t-j-1}U\tau\right)\\
&+ \int_0^1 P'_{i,T}\left([xT]\right)\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\Delta Y_{t-j} - E\left[\Delta Y_1\right]\right)\\
&\times \frac{1}{T}\sum_{\tau=1}^{T}\left(\frac{1}{\sqrt{T}}\sum_{\tau=1}^{t-j-1}U\tau\right)dx = O_p(1)
\end{aligned}
$$

where the last conclusion follows from Lemma A.1 and the fact that

$$
\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\Delta Y_{t-j} - E\left[\Delta Y_1\right]\right) = O_p(1)
$$

and

$$
\frac{1}{T}\sum_{\tau=1}^{T}\frac{1}{\sqrt{T}}\sum_{\tau=1}^{t-j-1}U\tau = O_p(1).
$$

Thus,

$$
\left(\overline{\mu}'_\perp \otimes I_{m+1}\right)\left(\beta'_\perp \otimes I_{m+1}\right)\frac{1}{T}\sum_{t=1}^{T}\overline{Y}_{t-1}^{(m)}\overline{X}'_t = O_p(1) \tag{51}
$$

Next, observe from (40) that

$$
\begin{aligned}
&\left(\overline{\mu}' \otimes I_{m+1}\right)\left(\beta'_\perp \otimes I_{m+1}\right)\frac{1}{T\sqrt{T}}\sum_{t=1}^{T}Y_{t-1}^{(m)}\overline{X}'_t\\
&\quad = \frac{1}{T\sqrt{T}}\sum_{t=1}^{T}P_T(t)\otimes\left(\overline{\mu}'\left(\alpha'_\perp\Omega\alpha_\perp\right)^{-1/2}\alpha'_\perp C'_0\sum_{j=1}^{t-1}U_j\right)\overline{X}'_t\\
&\quad + \frac{1}{T\sqrt{T}}\sum_{t=1}^{T}P_T(t)\otimes(t-1)\overline{X}'_t\\
&\quad + \frac{1}{T\sqrt{T}}\sum_{t=1}^{T}P_T(t)\otimes\left(\overline{\mu}'\beta'_\perp V_{t-1}\overline{X}'_t\right)\\
&\quad + \frac{1}{T\sqrt{T}}\sum_{t=1}^{T}P_T(t)\left(\overline{\mu}'\beta'_\perp\left(Y_0 - V_0\right)\overline{X}'_t\right)\\
&\quad = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}P_T(t)\otimes\left(\frac{t-1}{T}\right)\overline{X}'_t + O_p\left(1/\sqrt{T}\right)
\end{aligned}
$$

where the $O_p\left(1/\sqrt{T}\right)$ term is due similar to (51). Again, the latter matrix has typical element

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}P_{i,T}(t)\otimes\left(\frac{t-1}{T}\right)\left(\Delta Y_{t-j}-\frac{1}{T}\sum_{\tau=1}^{T}\Delta Y_{\tau-j}\right) \tag{52}$$

$$\begin{aligned}
&=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}P_{i,T}(t)\left(\frac{t-1}{T}\right)\left(\Delta Y_{t-j}-E\left[\Delta Y_1\right]\right)\\
&-\left(\frac{1}{T}\sum_{t=1}^{T}P_{i,T}(t)\left(\frac{t-1}{T}\right)\right)\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\Delta Y_{t-j}-E\left[\Delta Y_1\right]\right)\right)\\
&=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}P_{i,T}(t)\left(\frac{t-1}{T}\right)\left(\Delta Y_{t-j}-E\left[\Delta Y_1\right]\right)+O_p(1)\\
&=O_p(1)
\end{aligned}$$

where the last conclusion follows from Lemma A.2.

Finally, observe that

$$\begin{aligned}
\left(\beta'\otimes I_{m+1}\right)\frac{1}{T}\sum_{t=1}^{T}\overline{Y}_{t-1}^{(m)}\overline{X}_t' &= \frac{1}{T}\sum_{t=1}^{T}P_T(t)\otimes\left(\beta' V_{t-1}\overline{X}_t'\right) \\
&= \frac{1}{T}\sum_{t=1}^{T}P_T(t)\otimes\beta' V_{t-1}X_t' \\
&\quad -\left(\frac{1}{T}\sum_{t=1}^{T}P_T(t)\beta' V_{t-1}\right)\left(\frac{1}{T}\sum_{t=1}^{T}X_t'\right) \\
&= \left(\frac{1}{T}\sum_{t=1}^{T}P_T(t)\right)\otimes\beta' E\left[V_0X_1'\right]+o_p(1) \\
&= \begin{pmatrix}\beta' E\left[V_0X_1'\right]\\ O_{rm,k}\end{pmatrix}+o_p(1)
\end{aligned} \tag{53}$$

Thus, it follows from (51), (52) and (53) that

Lemma C.5. *Under Assumptions C.1-4,*

$$\left(M_T'\otimes I_{m+1}\right)\left(\beta_\perp'\otimes I_{m+1}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\overline{Y}_{t-1}^{(m)}\overline{X}_t\right)=O_p(1)$$

$$\left(\beta' \otimes I_{m+1}\right) \frac{1}{T} \sum_{t=1}^{T} \overline{Y}_{t-1}^{(m)} \overline{X}_t' = \begin{pmatrix} \beta' E\left[V_0 X_1'\right] \\ O_{rm,k} \end{pmatrix} + o_p(1)$$

3.6 Properties of the Matrix $S_{01,T}^{(m)}$

Note that we can rewrite (33) as

$$\triangle \overline{Y}_t = \alpha \beta' \overline{Y}_{t-1} + \Gamma_* \overline{X}_t + C_0 \overline{U}_t \tag{54}$$

where now $\Gamma_* = (\Gamma_1, ..., \Gamma_{p-1})$, $\overline{U}_t = U_i - \overline{U}$, where $\overline{U} = \frac{1}{T}\sum_{t=1}^{T} U_t$. Substituting (54) in (50) yields

$$\begin{aligned} S_{01,T}^{(m)} &= \alpha \frac{1}{T} \sum_{t=1}^{T} \beta' \overline{Y}_{t-1} \overline{Y}_{t-1}^{(m)\prime} \\ &-\alpha \left(\frac{1}{T} \sum_{t=1}^{T} \beta' \overline{Y}_{t-1} \overline{X}_t' \right) \left(\frac{1}{T} \sum_{t=1}^{T} \overline{X}_t \overline{X}_t' \right)^{-1} \left(\frac{1}{T} \sum_{t=1}^{T} \overline{X}_t \overline{Y}_{t-1}^{(m)\prime} \right) \\ &+C_0 \frac{1}{T} \sum_{t=1}^{T} \overline{U}_t \overline{Y}_{t-1}^{(m)\prime} \\ &-C_0 \left(\frac{1}{T} \sum_{t=1}^{T} \overline{U}_t \overline{X}_t' \right) \left(\frac{1}{T} \sum_{t=1}^{T} \overline{X}_t \overline{X}_t' \right)^{-1} \left(\frac{1}{T} \sum_{t=1}^{T} \overline{X}_t \overline{Y}_{t-1}^{(m)\prime} \right) \end{aligned} \tag{55}$$

hence

$$\begin{aligned} &\sqrt{T} \left(\alpha_\perp' \Omega \alpha_\perp\right)^{-1/2} \alpha_\perp' S_{01,T}^{(m)} \left(\beta \otimes I_{m+1}\right) \\ &= \left(\alpha_\perp' \Omega \alpha_\perp\right)^{-1/2} \alpha_\perp' C_0 \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \overline{U}_t \overline{Y}_{t-1}^{(m)\prime} \left(\beta \otimes I_{m+1}\right) \\ &- \left(\alpha_\perp' \Omega \alpha_\perp\right)^{-1/2} \alpha_\perp' C_0 \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \overline{U}_t \overline{X}_t' \right) \left(\frac{1}{T} \sum_{t=1}^{T} \overline{X}_t \overline{X}_t' \right)^{-1} \\ &\times \left(\frac{1}{T} \sum_{t=1}^{T} \overline{X}_t \overline{Y}_{t-1}^{(m)\prime} \left(\beta \otimes I_{m+1}\right) \right) \end{aligned} \tag{56}$$

$$= \left(\alpha_\perp' \Omega \alpha_\perp\right)^{-1/2} \alpha_\perp' C_0 \frac{1}{\sqrt{T}} \sum_{t=1}^{T} U_t \overline{Y}_{t-1}^{(m)'} \left(\beta \otimes I_{m+1}\right)$$

$$- \left(\alpha_\perp' \Omega \alpha_\perp\right)^{-1/2} \alpha_\perp' C_0 \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} U_t \overline{X}_t' \right) \left(\Sigma_{XX}^{-1} \Sigma_{X\beta}, O_{k(p-1),r.m} \right)$$

$$+ o_p(1)$$

$$= \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \vartheta_t \left(\overline{Y}_{t-1}^{(m)'} \left(\beta \otimes I_{m+1}\right) - \left(\overline{X}_t' \Sigma_{XX}^{-1} \Sigma_{X\beta}, 0_{r.m}' \right) \right) + o_p(1)$$

where

$$\Sigma_{XX} = p \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \overline{X}_t \overline{X}_t', \ \Sigma_{X\beta} = p \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \overline{X}_t \overline{Y}_{t-1}' \beta, \ \Sigma_{\beta X} = \Sigma_{X\beta}'$$

and

$$\vartheta_t = \left(\alpha_\perp' \Omega \alpha_\perp\right)^{-1/2} \alpha_\perp' C_0 U_t$$

It follows therefore similar to Lemma A.7 that

Lemma C.6. *Under Assumptions C.1-4,*

$$\left(\alpha_\perp' \Omega \alpha_\perp\right)^{-1/2} \alpha_\perp' S_{01,T}^{(m)} \left(\beta_\perp' \otimes I_{m+1}\right) \left(M_T \otimes I_{m+1}\right) \xrightarrow{d} \int_0^1 \left(dW_{k-r}\right) \widetilde{W}_{k-r,m}' \tag{57}$$

and

$$\sqrt{T} \left(\alpha_\perp' \Omega \alpha_\perp\right)^{-1/2} \alpha_\perp' S_{01,T}^{(m)} \left(\beta \otimes I_{m+1}\right) \xrightarrow{d} Z \tag{58}$$

jointly, where Z is a $(k-r) \times r(m+1)$ random matrix and

$$\begin{aligned} W_{k-r} &= \left(\alpha_\perp' \Omega \alpha_\perp\right)^{-1/2} \alpha_\perp' C_0 W \\ \widetilde{W}_{k-r,m}(x) &= p(x) \otimes \begin{pmatrix} \underline{W}_{k-r-1}(x) \\ x \end{pmatrix} \\ &\quad - \int_0^1 p(y) \otimes \begin{pmatrix} \underline{W}_{k-r-1}(y) \\ y \end{pmatrix} dy, \end{aligned} \tag{59}$$

with $\underline{W}_{k-r-1}$ defined by (42). In particular, the $k-r$ columns of Z' are independent

$$N_{r(m+1)} \left[0, \begin{pmatrix} \Sigma_{\beta\beta}^* & O_{r,r.m} \\ O_{r.m,r} & \Sigma_{\beta\beta} \otimes I_m \end{pmatrix} \right] \tag{60}$$

distributed, where

$$\Sigma_{\beta\beta} = p \lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \beta'\overline{Y}_{t-1}\overline{Y}'_{t-1}\beta,\ \Sigma^{*}_{\beta\beta} = \Sigma_{\beta\beta} - \Sigma_{\beta X}\Sigma^{-1}_{XX}\Sigma_{X\beta}$$

Moreover,

$$Z_2 = Z\begin{pmatrix} O_{r,r.m} \\ I_{r.m} \end{pmatrix} \tag{61}$$

is independent of $\widetilde{W}_{k-r,m}$ *and* W_{k-r}.

Proof: We only need to prove the last claim. Recall from (42) that $\underline{W}_{k-r-1} = \overline{\mu}'_{\perp}W_{k-r}$, so that we only need to show that Z_2 and W_{k-r} are independent. Note that

$$\begin{aligned}
&\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\vartheta_t\left(\overline{Y}^{(m)\prime}_{t-1}\left(\beta\otimes I_{m+1}\right)\begin{pmatrix} O_{r,r.m} \\ I_{r.m} \end{pmatrix}\right) \\
&= \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\vartheta_t\left(\left(Y^{(m)\prime}_{t-1}\left(\beta\otimes I_{m+1}\right) - E\left[Y^{(m)\prime}_{t-1}\left(\beta\otimes I_{m+1}\right)\right]\right)\begin{pmatrix} O_{r,r.m} \\ I_{r.m} \end{pmatrix}\right) \\
&-\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\vartheta_t\right)\frac{1}{T}\sum_{t=1}^{T}\left(Y^{(m)\prime}_{t-1}\left(\beta\otimes I_{m+1}\right) - E\left[Y^{(m)\prime}_{t-1}\left(\beta\otimes I_{m+1}\right)\right]\right)\begin{pmatrix} O_{r,r.m} \\ I_{r.m} \end{pmatrix} \\
&= \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\vartheta_t\left(\left(Y^{(m)\prime}_{t-1}\left(\beta\otimes I_{m+1}\right) - E\left[Y^{(m)\prime}_{t-1}\left(\beta\otimes I_{m+1}\right)\right]\right)\begin{pmatrix} O_{r,r.m} \\ I_{r.m} \end{pmatrix}\right) \\
&+o_p(1) \xrightarrow{d} Z_2
\end{aligned}$$

Consider the empirical processes

$$W_{T,k-r}(x) = \frac{1}{\sqrt{T}}\sum_{t=1}^{[xT]}\vartheta_t,\ Z_{2,T}(x) = \frac{1}{\sqrt{T}}\sum_{t=1}^{[xT]}\vartheta_t G_{t-1}$$

where

$$G_{t-1} = \left(Y^{(m)\prime}_{t-1}\left(\beta\otimes I_{m+1}\right) - E\left[Y^{(m)\prime}_{t-1}\left(\beta\otimes I_{m+1}\right)\right]\right)\begin{pmatrix} O_{r,r.m} \\ I_{r.m} \end{pmatrix},$$

and note that $W_{T,k-r} \Rightarrow W_{k-r}$. Then

$$\begin{aligned} E\left[Z_{2T}(x)W_{T,k-r}(y)'\right] &= \frac{1}{T}\sum_{t=1}^{[\min(x,y)T]} E\left[\vartheta_t G_{t-1}\vartheta_t'\right] \\ &= \frac{1}{T}\sum_{t=1}^{[\min(x,y)T]} E\left[\vartheta_t \left(E\left[G_{t-1}\right|\vartheta_t\right]\right)\vartheta_t'\right] \\ &= O \end{aligned}$$

because $E\left[G_{t-1}\right|\vartheta_t] = E\left[G_{t-1}\right] = 0$. This proves that Z_2 and W_{k-r} are independent.

3.7 Conclusion

It is easy to verify that Lemmas A.4 and A.5 carry over to the drift case. Therefore, if we redefine $\xi_{\perp,T}$ as

$$\xi_{\perp,T} = \left(\left(\beta_\perp \otimes I_{m+1}\right)\left(M_T \otimes I_{m+1}\right), \begin{pmatrix} O_{k,m.r} \\ \sqrt{T}\left(\beta\Sigma_{\beta\beta}^{-1/2} \otimes I_m\right) \end{pmatrix}\right)$$

and $\widetilde{W}_{k-r,m}$ as (59), then Lemmas 3-5 and Theorem 1 carry over.

4 Appendix D: Monte Carlo Results

4.1 Empirical Size

To check how close the asymptotic critical values based on the χ^2 distribution are to the ones based on the small sample null distribution, we have applied our test to 10,000 replications of the bivariate cointegrated vector time series process $Y_t = (Y_{1,t}, Y_{2,t})'$, where $Y_{1,t} = Y_{2,t} + U_{1,t}$, $Y_{2,t} = Y_{2,t-1} + U_{2,t}$ with $U_t = (U_{1,t}, U_{2,t})'$ drawn independently from the bivariate standard normal distribution, for various values of T and m. The results are given in Table D.1. In each entry, $q_{1-\alpha}$ stands for the empirical $1-\alpha$ quantile. Thus, they are the empirical critical values. The values in parenthesis are the acceptance frequencies based on the χ^2 critical values. The case $T = 324$ is included because this the sample size of the empirical application in section 6.

Table D.1: Empirical Distribution of the LR TVC Statistic

		$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=10$	$m=15$	$m=25$
$T=100$	$q_{0.90}$	5.320 (0.930)	9.195 (0.943)	12.787 (0.953)	16.320 (0.961)	19.854 (0.969)	38.759 (0.992)	60.610 (0.999)	119.797 (0.999)
	$q_{0.95}$	7.027 (0.970)	11.159 (0.975)	15.111 (0.980)	18.829 (0.984)	22.643 (0.987)	42.543 (0.997)	65.650 (0.999)	127.696 (0.999)
	$q_{0.99}$	10.426 (0.994)	15.271 (0.995)	19.973 (0.997)	24.160 (0.997)	28.643 (0.998)	49.833 (0.999)	76.269 (0.999)	143.749 (0.999)
$T=200$	$q_{0.9}$	4.880 (0.912)	8.313 (0.919)	11.607 (0.928)	14.693 (0.934)	17.792 (0.941)	33.273 (0.968)	49.068 (0.984)	85.794 (0.998)
	$q_{0.95}$	6.406 (0.959)	10.065 (0.960)	13.595 (0.965)	16.896 (0.968)	20.395 (0.974)	37.000 (0.988)	53.424 (0.994)	91.731 (0.999)
	$q_{0.99}$	9.666 (0.992)	14.188 (0.993)	17.834 (0.993)	21.993 (0.995)	25.364 (0.995)	43.754 (0.998)	62.006 (0.999)	102.433 (0.999)
$T=324$	$q_{0.90}$	4.790 (0.908)	8.149 (0.913)	11.181 (0.917)	14.059 (0.919)	17.050 (0.926)	31.247 (0.947)	45.621 (0.966)	76.331 (0.990)
	$q_{0.95}$	6.275 (0.956)	10.015 (0.959)	13.197 (0.959)	16.400 (0.963)	19.452 (0.965)	34.608 (0.977)	49.515 (0.986)	81.177 (0.996)
	$q_{0.99}$	9.530 (0.991)	14.173 (0.993)	18.042 (0.993)	21.193 (0.993)	24.749 (0.994)	40.850 (0.996)	56.899 (0.997)	91.638 (0.999)
$T=500$	$q_{0.90}$	4.658 (0.902)	7.945 (0.906)	10.952 (0.910)	13.861 (0.914)	16.616 (0.916)	30.083 (0.931)	43.651 (0.948)	71.478 (0.975)
	$q_{0.95}$	6.088 (0.952)	9.709 (0.954)	13.119 (0.958)	16.138 (0.959)	19.075 (0.960)	33.377 (0.969)	47.753 (0.979)	76.213 (0.990)
	$q_{0.99}$	9.092 (0.989)	13.898 (0.992)	17.313 (0.991)	20.767 (0.992)	24.063 (0.992)	39.983 (0.994)	55.072 (0.996)	84.851 (0.998)

4.2 Size and Power of the Park-Hahn Tests

Park and Hahn (1999) propose two types of test for TV cointegration, with statistics given by

$$\widehat{\tau}_1 = \frac{\sum_{t=1}^T \widehat{u}_t^2 - \sum_{t=1}^T \widehat{s}_t^2}{\widehat{\omega}_{T\kappa}^2},\ \widehat{\tau}_2 = \frac{\sum_{t=1}^T \left(\sum_{i=1}^t \widehat{u}_i\right)^2}{T^2 \widehat{\omega}_{T\kappa}^2},$$

where the $\widehat{u}_t$'s are the residuals of the regression of $Z_{1,t}$ on $Z_{2,t}$, the $\widehat{s}_t$'s are the residuals of the regression of $Z_{1,t}$ on $Z_{2,t}$ and t, $t^2, .., t^s$, and $\widehat{\omega}_{T\kappa}^2 = \frac{1}{T}\sum_{|k|<\ell_T} g\left(k/\ell_T\right) \sum_{t=k+1}^T \widehat{u}_{\kappa,t}\widehat{u}_{\kappa,t-k}$ is a long-run variance estimator, where the $\widehat{u}_{\kappa,t}$'s are the residuals of the regression of $Z_{1,t}$ on $\varphi_i\left(t/T\right) Z_{2,t}$ for $i = 1, ..., K$, with the φ_i's Fourier and other functions. As to the latter, we consider two cases, indicated by c:

$$c = 1 : \varphi_1(r) \ = \ \cos(2\pi r),\ \varphi_2(r) = \sin(2\pi r),\ \varphi_3(r) = 1,\ \varphi_4(r) = r$$

$$\begin{aligned} c = 2 : \varphi_1(r) &= \cos(2\pi r),\ \varphi_2(r) = \sin(2\pi r),\ \varphi_3(r) = \cos(4\pi r), \\ \varphi_4(r) &= \sin(4\pi r),\ \varphi_5(r) = 1,\ \varphi_6(r) = r,\ \varphi_7(r) = r^2 \end{aligned}$$

The statistic $\widehat{\tau}_1$ also depends on the polynomial order s. We consider the cases $s = 1$ and $s = 4$. Note that the test $\widehat{\tau}_2$ is in essence the well-known KPSS test. See Kwiatkowski et al. (1992). Finally, we use for g the Bartlett kernel, the truncation lag is $\ell_T = [T^{1/3}]$, the number of replications is $10,000$, and $T = 100, 200$. The results are presented in Table D.2.

Surprisingly, both Park-Hahn tests suffer from extreme size distortion, in particular the KPSS test $\widehat{\tau}_2$. Therefore, it is difficult to compare the actual power of these tests with the power of our test. Also, the size and power of test $\widehat{\tau}_1$ is quite sensitive to the choice of the functions φ_i's and the polynomial order s.

Table D.2: Size and power of the Park-Hahn tests

$\alpha_{asy} = 0.05$		$\widehat{\tau}_1$		$\widehat{\tau}_2$	
$T = 100$	$s = 4, c = 2$	$s = 1, c = 2$	$s = 4, c = 1$	$c = 2$	$c = 1$
$\omega = 0$	0.466	0.255	0.401	0.757	0.717
$\omega = 0.01$	0.481	0.277	0.419	0.766	0.728
$\omega = 0.05$	0.558	0.365	0.502	0.812	0.781
$\omega = 0.1$	0.669	0.453	0.623	0.864	0.840
$\omega = 0.2$	0.811	0.596	0.783	0.926	0.912
$\omega = 0.5$	0.957	0.786	0.950	0.990	0.988
$\omega = 1$	0.993	0.877	0.991	0.999	0.999
$T = 200$					
$\omega = 0$	0.391	0.211	0.353	0.696	0.672
$\omega = 0.01$	0.427	0.252	0.389	0.719	0.699
$\omega = 0.05$	0.613	0.411	0.580	0.825	0.807
$\omega = 0.1$	0.777	0.552	0.758	0.900	0.892
$\omega = 0.2$	0.920	0.724	0.912	0.971	0.967
$\omega = 0.5$	0.992	0.880	0.991	0.999	0.998
$\omega = 1$	0.998	0.936	0.998	1.000	1.000

5 Appendix E: Empirical Application

5.1 Test results

The asymptotic p-values[3] of our test are presented in Table E.1 for different combinations of the order m of the Chebyshev polynomial and the lag order p. Because the results did not vary much with p we only report the results for $p = 1$, $p = 6$, $p = 10$ (Falk and Wang's lag), and $p = 18$. Moreover, we have computed the p-values for m ranging from 1 to 25, although we only present relevant p-values.

Table E.1: Test results for the PPP hypothesis

	m	$p=1$	$p=6$	$p=10$	$p=18$
Can	≥ 1	0.000	0.000	0.000	0.000
Fra	≥ 1	0.000	0.000	0.000	0.000
Ger	1	0.158	0.461	0.033	0.010
	≥ 2	0.000	0.000	0.000	0.000
Ita	≥ 1	0.000	0.000	0.000	0.000
Jap	1	0.662	0.383	0.001	0.019
	2	0.102	0.457	0.001	0.004
	3	0.029	0.013	0.000	0.000
	4	0.037	0.014	0.000	0.000
	≥ 5	0.000	0.000	0.000	0.000
U.K.	1	0.119	0.053	0.242	0.023
	2	0.194	0.039	0.011	0.001
	3	0.004	0.000	0.000	0.000
	≥ 4	0.000	0.000	0.000	0.000

5.2 Plots of the TV Cointegrating Vectors

At the end of this Appendix we present the plots of the time-varying coefficients β_{1t}, β_{2t} and β_{3t} in the cointegrating PPP relation $\beta_t' Y_t = \beta_{1t} \ln S_t^f + \beta_{2t} \ln P_t^n + \beta_{3t} \ln P_t^f$, where P_t^n and P_t^f are the price indices in the domestic and foreign economies, respectively, and S_t^f is the nominal exchange rate in home currency per unit of the foreign currency. The Chebyshev polynomial order m was chosen according to the Hannan-Quinn criterion, and the VECM order p was chosen $p = 1$.

[3] $P\left(\chi^2_{(mrk)} \geq LR^{tvc}\right)$.

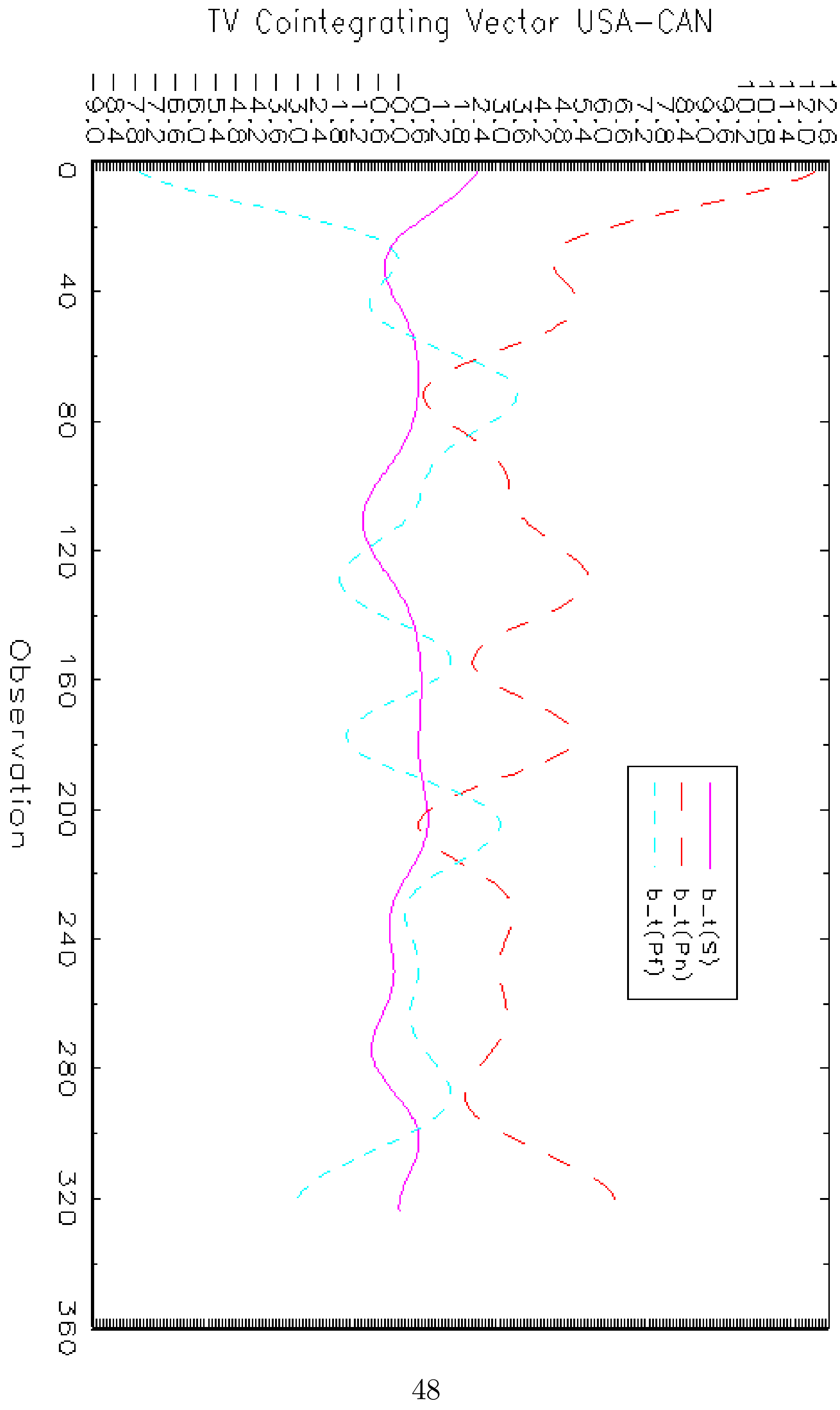

TV Cointegrating Vector USA-CAN
Observation
b_t(s)
b_t(Pn)
b_t(Pf)
0
40
80
120
160
200
240
280
320
360

Figure 1:

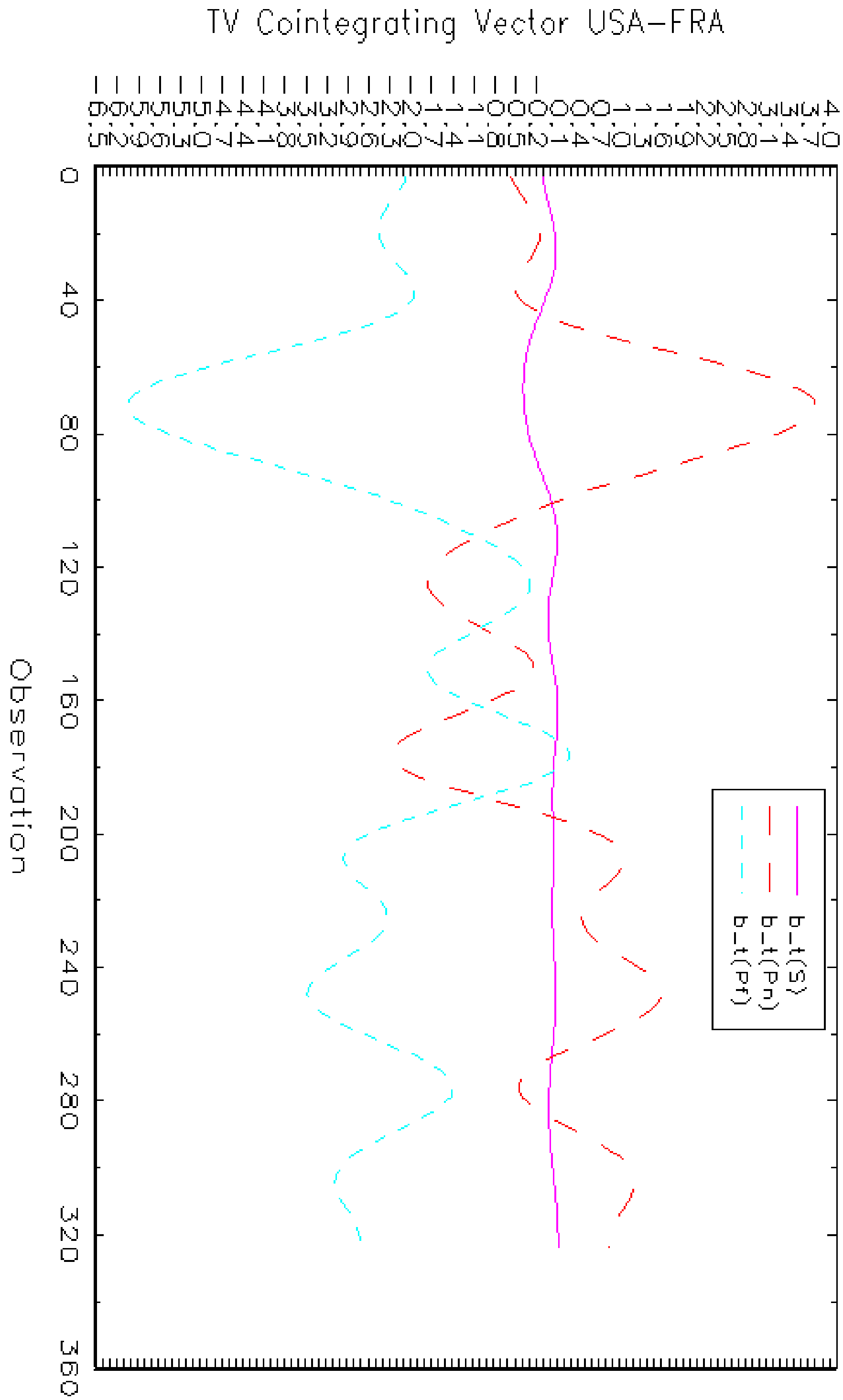

TV Cointegrating Vector USA-FRA
Observation
0
40
80
120
160
200
240
280
320
360
b_t(S)
b_t(Pn)
b_t(Pf)

Figure 2:

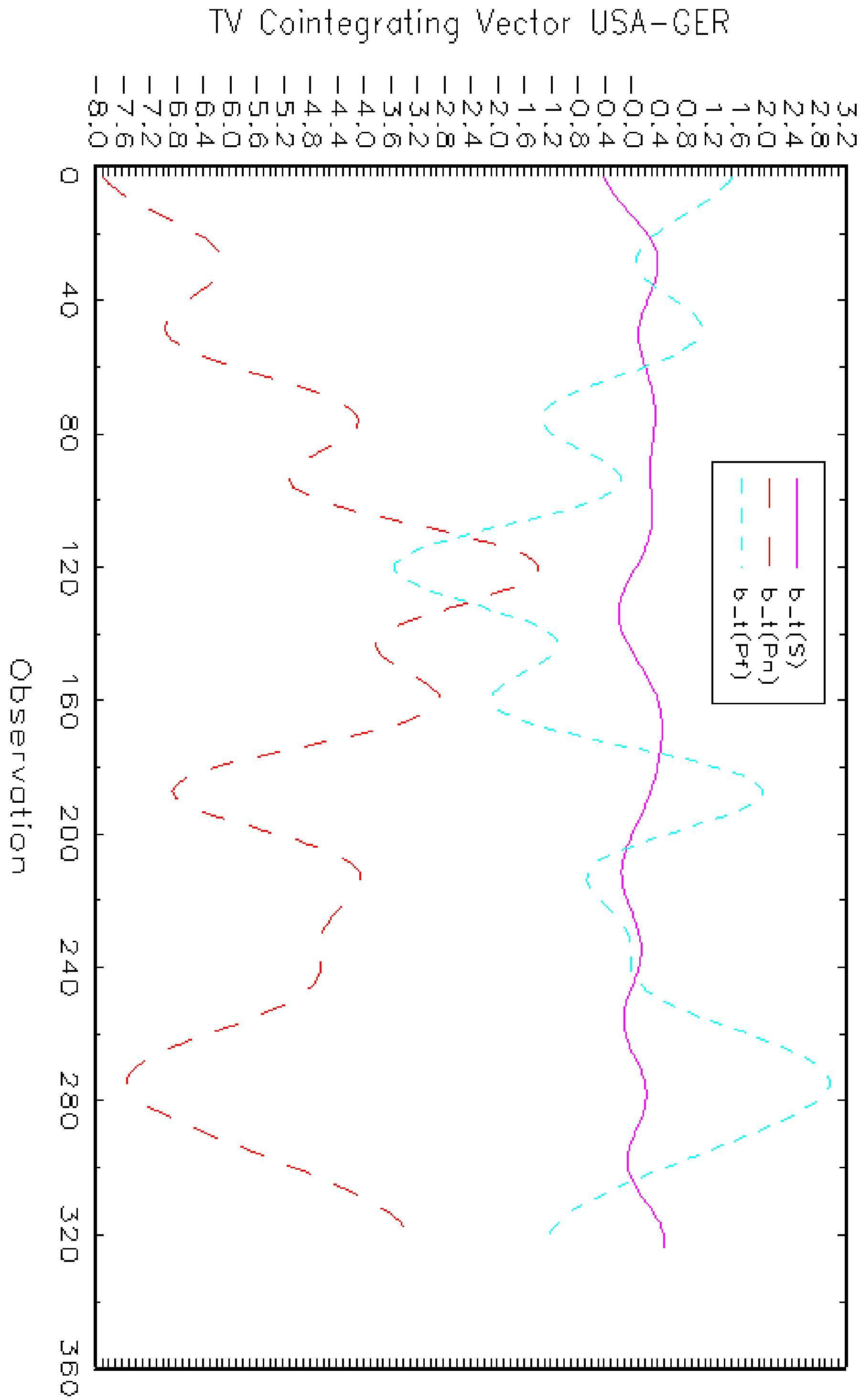
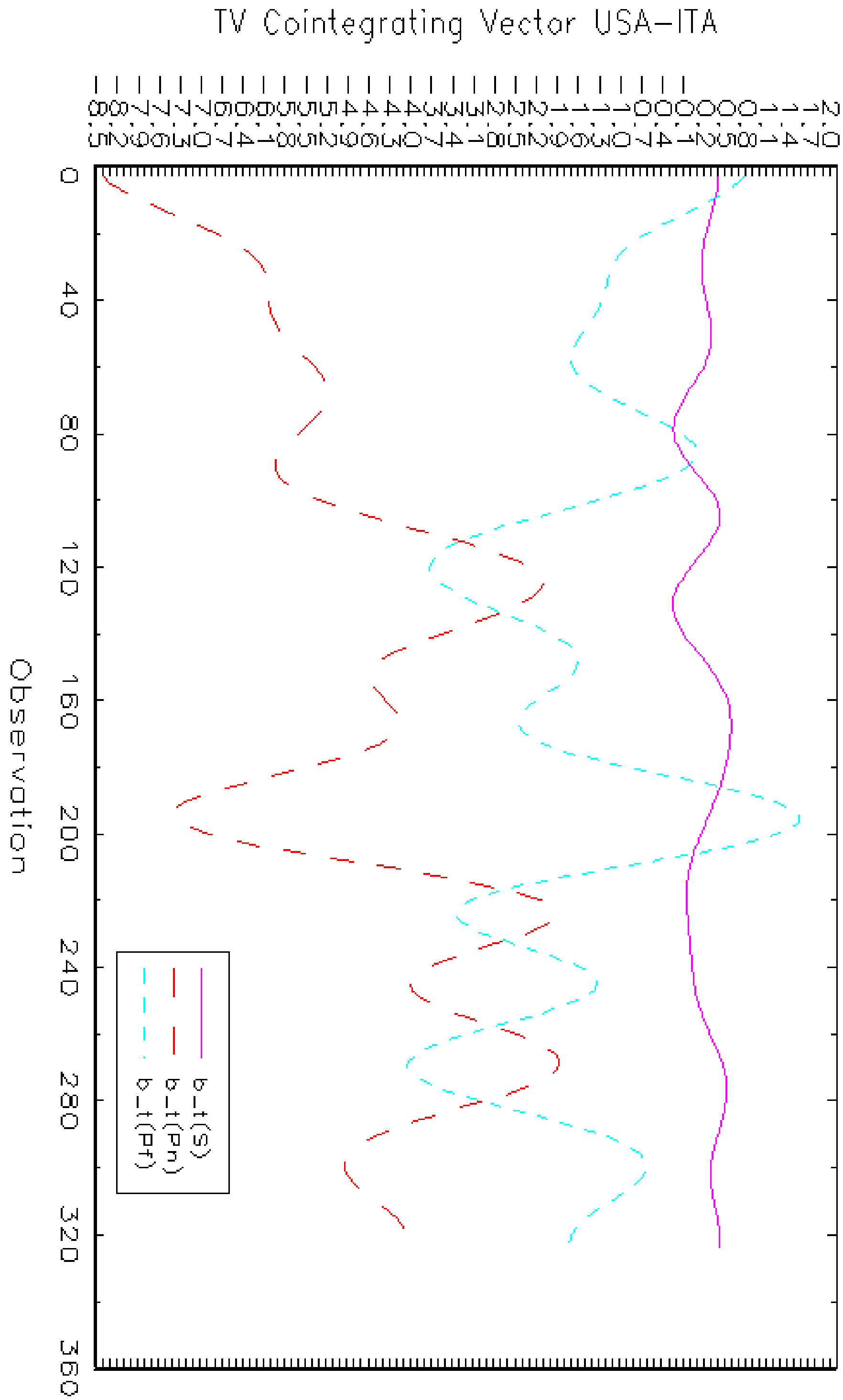
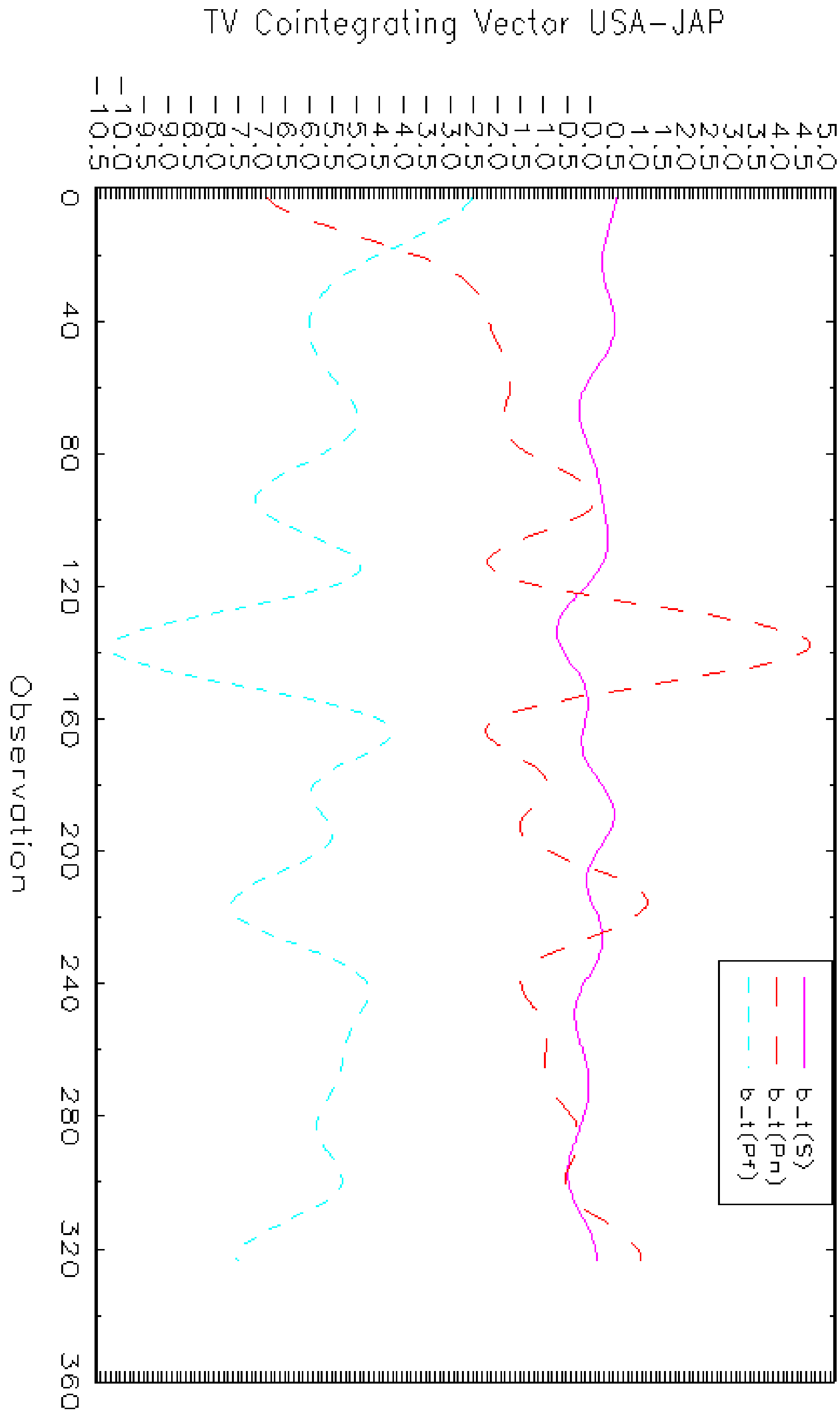


Figure 3:



TV Cointegrating Vector USA-ITA
Observation
0
40
80
120
160
200
240
280
320
360
b_t(S)
b_t(Pn)
b_t(Pf)

Figure 4:


TV Cointegrating Vector USA-JAP
Observation
0
40
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160
200
240
280
320
360
b_t(S)
b_t(Pn)
b_t(Pf)

Figure 5:

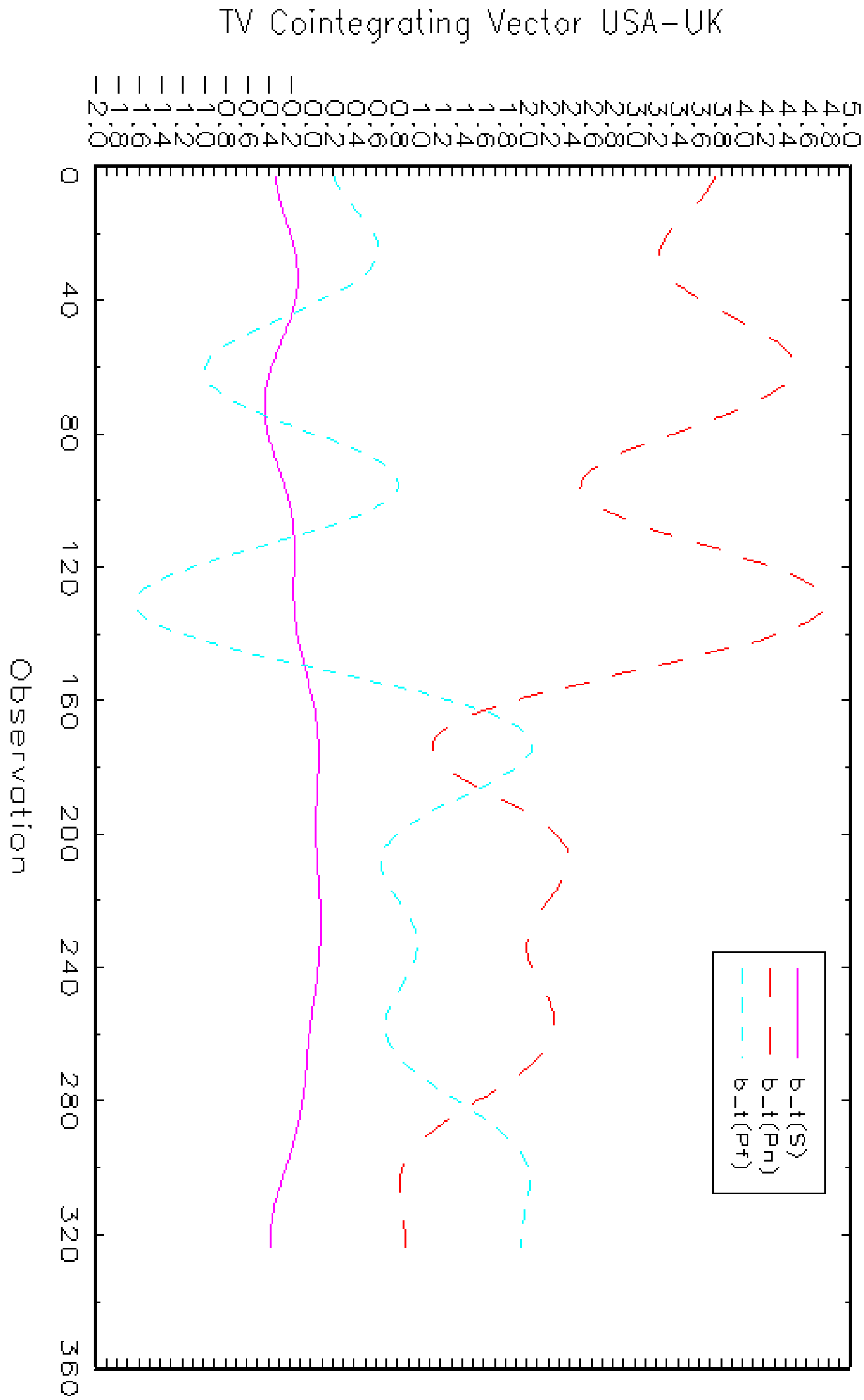
TV Cointegrating Vector USA-UK
Observation
0
40
80
120
160
200
240
280
320
360
b_t(S)
b_t(Pn)
b_t(Pf)

Figure 6:

6 Additional References

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