Valuation of Interest Rate Path Dependent Options under Multi-Factor Gaussian HJM Models

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Abstract

Interest rate caps and floors with barrier features are proposed and valued, in the context of a multi-factor, not necessarily Markovian, Gaussian HJM framework, and under the assumption of continuous monitoring of the barrier. The analytical pricing solutions derived for such path-dependent interest rate options are subject to a proportionality assumption similar to the *rank 1* approximation suggested by Brace and Musiela (1994a). Nevertheless, the resulting approximations are shown to be much faster than the existing exact numerical methods, as well as accurate.

The proposed pricing methodology is also extended to the valuation of lookback interest rate options.

Key words: Gaussian HJM models, Time-change, Change of probability measure, Bivariate normal distribution, Single-barrier, Partial-barrier, and Lookback caps and floors.

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1 Introduction

The main purpose of this paper is to derive analytical valuation formulas for *single-barrier*, *partial-barrier* and (fixed-strike as well as floating-strike) *lookback* interest rate caps and floors, in the context of a multi-factor Gaussian Heath, Jarrow and Morton (1992a) term structure model and under continuous monitoring of the underlying interest rate.

As for regular caps (floors), barrier caps (floors) are also portfolios of caplets (floorlets), that is, sequences of European call (put) options on nominal -i.e., discretely (as opposed to continuously) compounded- "money-market" interest rates (for instance, on the 3-month US Libor). However, the component barrier caplets (or barrier floorlets) only generate their regular terminal payoff (with settlement in arrears) if, during some time-period, the barrier is (*knock-in options*) or is not (*knock-out options*) touched by the underlying nominal spot interest rate. For *single-barrier* caps or floors, the barrier is monitored during the all life of each component caplet or floorlet. For *partial-barrier* caps or floors, the barrier is monitored only during a part of the life of each component caplet or floorlet. In any case, the existence of a barrier rate makes such interest rate options cheaper than their regular counterparts as well as more easily tailored to the investor' expectations about the evolution of the underlying interest rate.

Lookback caps (floors) are again sequences of European options on nominal "money-market" interest rates, but its terminal payoff (with settlement in arrears) will depend on the maximum or on the minimum attained by the underlying spot nominal interest rate over a certain period of time. *Floating strike lookback* caplets (floorlets) are European call (put) options on a spot nominal interest rate, but with a strike equal to the lowest (highest) level registered by the underlying during the option' life. *Fixed strike lookback* caplets (floorlets) are European call (put) options on the maximum (minimum) attained by a spot nominal interest rate during the option' life, that is are European options on the extrema of a "money-market" spot interest rate.

Although barrier and lookback options are amongst the oldest and most popular forms of exotic options,[1] the extensive treatment that they have received in the Finance literature -see, for instance, Merton (1973), Rubinstein and Reiner (1991), Heynen and Kat (1994), Rich (1994), Goldman, Sosin and Gatto (1979), and Conze and Viswanathan (1991)- has been almost always confined to the case where the underlying asset price follows a geometric Brownian motion process (that is, typically, to barrier options on stocks, indices or currencies). To the author's knowledge, such path-dependent option pricing theory has not been applied yet to the valuation of OTC exotic interest rate caps and swaptions, which are commonly named as representing the largest (in terms of notional amount) market of interest rate options. This paper is intended to provide a first contribution towards this direction, by proposing and valuing the above mentioned path-dependent interest rate contracts.

The main difficulty in valuing barrier or lookback caps and floors arises from the fact that the underlying for such path-dependent interest rate options is not a forward nominal interest rate -as it can be thought to be the case for regular caps or floors- but, instead, a spot nominal interest rate, which is directly observed in the market. Such contract feature was also essential in choosing the valuation framework. Because the lognormal property of forward rates is not preserved for spot rates, under a Libor Market model, a Gaussian -but not necessarily Markovian- HJM setup is used instead, for reasons of analytical tractability. Despite the underlying restrictive assumption of a deterministic -although, possibly, time-inhomogeneous- specification for the volatility function, the model is still consistent with the initially observed term structure of interest rates. Moreover, a multi-factor formulation is adopted in order to potentiate the model' calibration to the market covariance matrix, as suggested by Rebonato (1998, page 70).

Next sections are organized as follows. Section 2 describes the multi-factor Gaussian HJM model that will be used to derive approximate pricing solutions for the new contracts proposed in this paper. Section 3 describes and prices *single-barrier* caps and floors. For that purpose, transition densities conditional on touching or not the barrier, and for an appropriate martingale probability measure, are derived in theorem 1. Then, section 4 converts all the previous results for *partial-barrier* caps and floors. Section 5 derives probability density functions for the extreme values of the underlying spot nominal interest rate and provides analytical approximate pricing solutions for *lookback* caps and floors. Finally, section 6 summarizes the main results and provides some directions for further research. All accessory proves are relegated to the appendix,

[1] See, for instance, the historical notes contained in Zhang (1998, page 203).

while the more illustrative ones are kept in the text. Numerical examples are also presented in order to test the accuracy of the proposed pricing solutions.

2 Model description

Hereafter, $\mathcal{Q}$ will denote the martingale probability measure obtained when the "money market account" is taken as the numeraire of the economy underlying the model under analysis.[2] In such underlying stochastic intertemporal economy there exists a trading interval $\mathcal{T} = [t_0, \tau]$, for some fixed time $\tau > t_0$, and uncertainty is represented by a probability space $(\Omega, \mathcal{F}, \mathcal{Q})$, where all the information accruing to all the agents in the economy is described by a filtration $(\mathcal{F}_t)_{t \in \mathcal{T}}$ satisfying the usual conditions: namely, $\mathcal{F}_{t_0}$ is assumed to be almost trivial, and $\mathcal{F}_\tau = \mathcal{F}$.

It is further assumed that there exists an arbitrage-free[3] and frictionless market for pure discount bond prices, which are considered to be perfectly divisible and to evolve through time according to the following stochastic differential equation:

$$\frac{dP(t,T)}{P(t,T)} = r(t)\,dt + \underline{\sigma}(t,T)' \cdot d\underline{W}^{\mathcal{Q}}(t), \tag{1}$$

where $P(t,T)$ represents the time-t price of a (unit face value) zero coupon bond expiring at time T, for all $T \in [t_0, \tau]$ and $t \in [t_0, T]$, $r(t)$ is the time-t instantaneous spot rate, which can be defined by continuity as

$$r(t) := \lim_{T \to t} \left[-\frac{\ln P(t,T)}{T-t} \right],$$

$\cdot$ denotes the inner product in $\Re^n$, $:=$ means equal by definition, and $\underline{W}^{\mathcal{Q}}(t) \in \Re^n$ is a n-dimensional and $\mathcal{Q}$ -measurable standard Brownian motion, initialized at zero and generating the augmented, right continuous and complete filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq t_0\}$. The n-dimensional adapted volatility function $\underline{\sigma}(\cdot, T) : [t_0, T] \to \Re^n$ is assumed to satisfy the usual mild measurability and integrability requirements -as stated, for instance, in Lamberton and Lapeyre (1996, theorem 3.5.5)- as well as the boundary condition[4] $\underline{\sigma}(u,u) = \underline{0} \in \Re^n, \forall u \in [t_0, T]$. Moreover, for reasons of analytical tractability, that is in order to obtain lognormally distributed pure discount bond prices, the following modelling restriction is adopted.

Assumption 1 The volatility function $\underline{\sigma}(\cdot, T) : [t_0, T] \to \Re^n$ is deterministic.

Remark 1 *Of course, and as shown by equation (4), the main theoretical model' limitation that arises from assumption 1 is the normality of forward rates, that is, the fact that interest rates can attain negative values with positive probability.*[5] *However, the possibility of generalizing the pricing solutions derived in the next sections for more complex volatility structures (precluding the existence of negative forward rates) -using, for instance, the approach suggested by Nunes, Clewlow and Hodges (1999)- is still an open question for further research.*

Equation (1), equipped with assumption 1, represents the *Gaussian* interest rate term structure model that will be used to derive analytical pricing solutions for European path-dependent interest rate options. Such model can be rewritten in terms of instantaneous forward interest rates, that is under the usual Heath et al. (1992a) specification. Applying Itô's lemma to equation (1), it follows that

$$\ln P(t,T) = \ln P(t_0,T) + \int_{t_0}^{t} \left[r(s) - \frac{1}{2}\underline{\sigma}(s,T)' \cdot \underline{\sigma}(s,T) \right] ds + \int_{t_0}^{t} \underline{\sigma}(s,T)' \cdot d\underline{W}^{\mathcal{Q}}(s). \tag{2}$$

[2] Meaning that the relative prices of all assets with respect to the numeraire given by a "money market account" are $\mathcal{Q}$-martingales.

[3] Following Harrison and Pliska (1981), the absence of arbitrage is implied by the existence of the equivalent martingale measure $\mathcal{Q}$.

[4] Consistently with the "pull-to-par" phenomena.

[5] Rogers (1996) identifies the potential pricing implications of such model "deficiency", even for small (risk-neutral) probabilities of attaining negative interest rates.

Then, using the definition of the time-t instantaneous forward rate for date $T > t$,

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T}, \tag{3}$$

and assuming that $\frac{\partial \ln P(t,T)}{\partial T}$ is well defined, the usual Gaussian HJM model' specification arises:

$$f(t,T) = f(t_0,T) + \int_{t_0}^{t} \frac{\partial \underline{\sigma}(s,T)'}{\partial T} \cdot \underline{\sigma}(s,T)\, du - \int_{t_0}^{t} \frac{\partial \underline{\sigma}(s,T)'}{\partial T} \cdot d\underline{W}^{\mathbb{Q}}(s). \tag{4}$$

Remark 2 *Model (4) is clearly arbitrage-free in the sense that the drift of $df(t,T)$ satisfies the usual HJM no-arbitrage condition. Moreover, the model is consistent with the initially observed forward rate curve $\{f(t_0,T) : T \in [t_0,\tau]\}$, and completely specified by the volatility function $\underline{\sigma}(\cdot,T) : [t_0,T] \to \Re^n$.*

Remark 3 *Since model (4) will be used to price (path-dependent) options on Libor rates, that is on interbank (risky) rates, instead of government (risk-free) rates, the Duffie and Singleton (1997) assumption of symmetric counterparty credit risk will be implicitly used hereafter. Meaning that the model' forward interest rates will be regarded not as riskless rates but rather as default- and liquidity-adjusted rates.*

3 Single-barrier caps and floors

3.1 Terminal payoffs

This paper proposes and values four types of single-barrier caps and floors, subject to continuous monitoring of the barrier, namely: *up-and-in caps*, *down-and-out caps*, *down-and-in floors*, and *up-and-out floors*. All these new path-dependent interest rate options share a common appealing feature that makes them cheaper when compared with their regular vanilla counterparts: they are only activated if the underlying spot rate evolves in a favorable direction; or, they terminate their lives worthless if the underlying spot rate evolves in an unfavorable direction.[6]

As for regular caps and floors, the fair price of a forward-start single-barrier cap or floor corresponds to the summation of the fair prices for all the component (single-barrier) caplets or floorlets. That is, the time-t_0 price of a $m\delta$-years forward-start single-barrier cap (floor) on a unitary principal, with a cap (floor) rate of r_k, subject to a barrier rate of r_H, with a rebate rate R, and settled in arrears at times $t_i = t_0 + i\delta$, $i = 2, \ldots, m$, is equal to[7]

$$\sum_{i=1}^{m-1} v_{t_0}\left[r_n(t_i, t_{i+1}); r_k; r_H; R; t_{i+1}\right], \tag{5}$$

where $v_{t_0}\left[r_n(t_i, t_{i+1}); r_k; r_H; R; t_{i+1}\right]$ represents the time-t_0 price of a generic single-barrier and unit face value caplet (floorlet) on the interest rate $r_n(t_i, t_{i+1})$, with a cap (floor) rate of r_k, subject to a barrier rate of r_H, with a rebate rate R, and settled in arrears at time t_{i+1}. The interest rate $r_n(t_i, t_{i+1})$ represents the nominal time-t_i spot rate for the compounding period $(t_{i+1} - t_i)$, and can be defined as

$$r_n(t_i, t_{i+1}) := \frac{1}{\delta}\left[\frac{1}{P(t_i, t_{i+1})} - 1\right]. \tag{6}$$

An *up-and-in caplet* generates the same terminal payoff as a regular caplet as long as the underlying spot nominal interest rate touches (from bellow) the barrier rate during the life of the option. Otherwise, a rebate (possibly null) is paid on the same terminal date. Similarly, a *down-and-in floorlet* generates the same terminal payoff as a regular floorlet, if the underlying spot nominal interest rate touches (from above) the barrier rate during the life of the option, or a rebate, otherwise. More formally,

[6] Four other types of single-barrier caps and floors are also feasible: *up-and-out caps*, *down-and-in caps*, *down-and-out floors*, and *up-and-in floors*. Although it would be simple to also price them explicitly, our main attention will be focussed on the first four more "intuitive" structures.

[7] Without loss of generality, and in order to simplify the notation, we shall assume that $t_{i+1} - t_i = \delta, \forall i$.

Definition 1 *The time-t_{i+1} price of a unit face value up-and-in caplet (down-and-in floorlet) on the nominal spot rate $r_n(t_i, t_{i+1})$, with a cap (floor) rate of r_k, subject to a barrier rate of r_u (r_l), with a rebate rate R, and settled in arrears at time t_{i+1} is equal to*[8]

$$IN(\theta)_{t_{i+1}}[r_n(t_i, t_{i+1}); r_k; r_H; R; t_{i+1}] := \begin{cases} [\theta r_n(t_i, t_{i+1}) - \theta r_k]^+ \delta \Longleftarrow \exists s \in]t_0, t_i] : \theta r_n(s, s+\delta) \geq \theta r_H \\ R\delta \Longleftarrow \theta r_n(s, s+\delta) < \theta r_H, \forall s \in]t_0, t_i] \end{cases}, \tag{7}$$

where $r_H = r_u (> r_n(t_0, t_0+\delta))$ and $\theta = 1$ for an up-and-in caplet, or $r_H = r_l (< r_n(t_0, t_0+\delta))$ and $\theta = -1$ for a down-and-in floorlet.

Remark 4 *Notice that the monitoring of the barrier is defined in terms of the observable nominal spot rate $r_n(s, s+\delta)$ and not in terms of the nominal forward rate $r_n(s, t_i, t_{i+1}) := \frac{1}{\delta}\left[\frac{P(s,t_i)}{P(t_i,t_{i+1})} - 1\right]$, which would be typically the underlying variable for a Libor Market model. This is the case because the latter interest rate is not directly observed in the money market.*[9] *Consequently, it is not possible to take advantage from the lognormal martingale formulation offered by a Market model, and, hence, an HJM model will be used instead.*

A *down-and-out caplet* generates the same terminal payoff as a regular caplet as long as the underlying spot nominal interest rate never touches (from above) the barrier rate during the life of the option. Otherwise, a rebate is paid on the same terminal date. Finally, an *up-and-out floorlet* generates the same terminal payoff as a regular floorlet, if the underlying spot nominal interest rate never touches (from bellow) the barrier rate during the life of the option, or a rebate, otherwise. That is,

Definition 2 *The time-t_{i+1} price of a unit face value down-and-out caplet (up-and-out floorlet) on the nominal spot rate $r_n(t_i, t_{i+1})$, with a cap (floor) rate of r_k, subject to a barrier rate of r_l (r_u), with a rebate rate R, and settled in arrears at time t_{i+1} is equal to*

$$OUT(\theta)_{t_{i+1}}[r_n(t_i, t_{i+1}); r_k; r_H; R; t_{i+1}] := \begin{cases} [\theta r_n(t_i, t_{i+1}) - \theta r_k]^+ \delta \Longleftarrow \theta r_n(s, s+\delta) > \theta r_H, \forall s \in]t_0, t_i] \\ R\delta \Longleftarrow \exists s \in]t_0, t_i] : \theta r_n(s, s+\delta) \leq \theta r_H \end{cases}, \tag{8}$$

where $r_H = r_l (< r_n(t_0, t_0+\delta))$ and $\theta = 1$ for a down-and-out caplet, or $r_H = r_u (> r_n(t_0, t_0+\delta))$ and $\theta = -1$ for an up-and-out floorlet.

Assumption 2 The rebate rate is zero.

Remark 5 *For simplicity, all the pricing solutions that will be derived hereafter assume that $R = 0$. Such analytical solutions can be easily generalized for $R \neq 0$, by considering the first passage time distribution of the underlying nominal spot rate.*

Because all the above mentioned options possess a single future payoff (at time t_{i+1}), their fair prices can be simply computed as the discounted expectation of such terminal payoffs, under the corresponding equivalent (forward) martingale probability measure.[10] Let $\mathcal{Q}^{t_{i+1}}$ be the t_{i+1}-forward martingale measure obtained when the t_{i+1}-maturity pure discount bond is taken as the numeraire of the economy. Such measure is equivalent to $\mathcal{Q}$, and will be defined on the same measurable space $(\Omega, \mathcal{F})$ through the following *Radon-Nikodym derivative*:

$$\left.\frac{d\mathcal{Q}^{t_{i+1}}}{d\mathcal{Q}}\right| \mathcal{F}_t := \frac{P(t, t_{i+1})}{P(t_0, t_{i+1})} \frac{\beta(t_0)}{\beta(t)}, \tag{9}$$

where $\beta(t)$ represents the time-t value of the "money market account", i.e. the compounded value of one monetary unit continuously reinvested, from time t_0 to time t, at the short-term interest rate:

$$\beta(t) := \exp\left[\int_{t_0}^{t} r(s)\, ds\right]. \tag{10}$$

[8] As usual, $[x]^+ = \max(x, 0), \forall x \in \Re$.

[9] For a *forward rate agreement* (FRA), the start and end dates of the underlying forward compounding period change every day (by moving forward one day). Unfortunately, for the contracts under valuation, t_i and t_{i+1} are fixed.

[10] See, for instance, Geman, Karoui and Rochet (1995).

Assuming that measure $Q^{t_{i+1}}$ exists, and following Harrison and Pliska (1981), the relative price, with respect to the t_{i+1}-maturity zero-coupon bond, of any attainable contingent claim that settles at time t_{i+1} will be a $Q^{t_{i+1}}$-martingale. Therefore, applying assumption 2 and relation (6) to definitions (7) and (8), the time-t_0 price of the above defined barrier options can be written not as a function of the nominal spot rate, but in terms of log-inverse pure discount prices:[11]

$$\begin{aligned} & IN(\theta)_{t_0}\left[r_n(t_i,t_{i+1});r_k;r_H;R=0;t_{i+1}\right] \qquad (11) \\ = \; & P(t_0,t_{i+1}) \\ & E_{t_0}^{Q^{t_{i+1}}}\left\{\left[\theta\exp\left(\ln P^{-1}(t_i,t_{i+1})\right)-\theta(1+\delta r_k)\right]^+ \middle| \sup_{t_0\le s\le t_i}\left[\theta\ln P^{-1}(s,s+\delta)\right]\ge\theta\ln(1+\delta r_H)\right\}, \end{aligned}$$

and

$$\begin{aligned} & OUT(\theta)_{t_0}\left[r_n(t_i,t_{i+1});r_k;r_H;R=0;t_{i+1}\right] \qquad (12) \\ = \; & P(t_0,t_{i+1}) \\ & E_{t_0}^{Q^{t_{i+1}}}\left\{\left[\theta\exp\left(\ln P^{-1}(t_i,t_{i+1})\right)-\theta(1+\delta r_k)\right]^+ \middle| \inf_{t_0\le s\le t_i}\left[\theta\ln P^{-1}(s,s+\delta)\right]>\theta\ln(1+\delta r_H)\right\}. \end{aligned}$$

To convert the last two equations into analytical pricing solutions, that is in order to compute explicitly both conditional expectations, it is necessary to derive the probability density function of terminal log-inverse pure discount bond prices with a time-to-maturity of δ-years, conditional on touching -for equation (11)- or on not touching -for equation (12)- the barrier, and under the t_{i+1}-forward martingale measure. Next, the corresponding unconditional distribution is firstly computed.

3.2 Unrestricted density for log-inverse pure discount bond prices

Combining equations (2) and (10), definition (9) can be restated as

$$\left.\frac{dQ^{t_{i+1}}}{dQ}\right|\mathcal{F}_t=\exp\left[\int_{t_0}^t\underline{\sigma}(s,t_{i+1})'\cdot d\underline{W}^Q(s)-\frac{1}{2}\int_{t_0}^t\underline{\sigma}(s,t_{i+1})'\cdot\underline{\sigma}(s,t_{i+1})\,ds\right].$$

Consequently, if measure $Q^{t_{i+1}}$ exits, that is if the following *Novikov's condition* is satisfied

$$E_{t_0}^{Q^{t_{i+1}}}\left\{\exp\left[\frac{1}{2}\int_{t_0}^t\underline{\sigma}(s,t_{i+1})'\cdot\underline{\sigma}(s,t_{i+1})\,ds\right]\right\}<\infty,$$

then *Girsanov's theorem* implies that

$$d\underline{W}^{Q^{t_{i+1}}}(t)=d\underline{W}^Q(t)-\underline{\sigma}(t,t_{i+1})\,dt \qquad (13)$$

is also a vector of standard Brownian motion increments in $\Re^n$ (with the same standard filtration as $d\underline{W}^Q(t)$). Hence, the Gaussian HJM model (4) can be re-specified under the equivalent measure $Q^{t_{i+1}}$ as

$$f(t,T)=f(t_0,T)+\int_{t_0}^t\frac{\partial\underline{\sigma}(s,T)'}{\partial T}\cdot\left[\underline{\sigma}(s,T)-\underline{\sigma}(s,t_{i+1})\right]ds-\int_{t_0}^t\frac{\partial\underline{\sigma}(s,T)'}{\partial T}\cdot d\underline{W}^{Q^{t_{i+1}}}(s), \qquad (14)$$

and t_{i+1}-forward martingale probability distributions can be easily obtained.

Proposition 1 *Under the equivalent martingale measure $Q^{t_{i+1}}$, for the Gaussian HJM specification (4), and conditional on $\mathcal{F}_{t_0}$, the random variable $\ln P^{-1}(t,t+\delta)$ possess a univariate normal distribution with*

[11] $E_{t_0}^{Q^{t_{i+1}}}(X|Y)$ denotes the expected value of the random variable X, conditional on the event Y and on $\mathcal{F}_{t_0}$, computed under the equivalent martingale measure $Q^{t_{i+1}}$. Notice that $\sup_{t_0\le s\le t_i}\left[-\ln P^{-1}(s,s+\delta)\right]=-\inf_{t_0\le s\le t_i}\left[\ln P^{-1}(s,s+\delta)\right]$ and $\inf_{t_0\le s\le t_i}\left[-\ln P^{-1}(s,s+\delta)\right]=-\sup_{t_0\le s\le t_i}\left[\ln P^{-1}(s,s+\delta)\right]$.

mean $\ln\left[\frac{P(t_0,t)}{P(t_0,t+\delta)}\right] + l(t_0,t)$ *and standard deviation* $\sqrt{g(t_0,t)}$*, that is*[12]

$$\Pr_{\mathcal{Q}^{t_{i+1}}}\left[\ln P^{-1}(t,t+\delta)\in dx\middle|\mathcal{F}_{t_0}\right] = \phi\left\{x;\ln\left[\frac{P(t_0,t)}{P(t_0,t+\delta)}\right] + l(t_0,t),\sqrt{g(t_0,t)}\right\}dx, \tag{15}$$

where[13]

$$l(t_0,t) := \int_{t_0}^{t}\left\{\frac{1}{2}\left[\|\underline{\sigma}(s,t+\delta)\|^2 - \|\underline{\sigma}(s,t)\|^2\right] - \left[\underline{\sigma}(s,t+\delta)-\underline{\sigma}(s,t)\right]'\cdot\underline{\sigma}(s,t_{i+1})\right\}ds, \tag{16}$$

and

$$g(t_0,t) := \int_{t_0}^{t}\|\underline{\sigma}(s,t)-\underline{\sigma}(s,t+\delta)\|^2\,ds. \tag{17}$$

Proof. Integrating equation (3) between t and $(t+\delta)$, and using equation (14),

$$\begin{aligned}\ln P^{-1}(t,t+\delta) &= \int_t^{t+\delta} f(t_0,u)\,du + \int_t^{t+\delta}\int_{t_0}^{t}\frac{\partial\underline{\sigma}(s,u)'}{\partial u}\cdot\left[\underline{\sigma}(s,u)-\underline{\sigma}(s,t_{i+1})\right]ds\,du \\ &\quad - \int_t^{t+\delta}\int_{t_0}^{t}\frac{\partial\underline{\sigma}(s,u)'}{\partial u}\cdot d\underline{W}^{\mathcal{Q}^{t_{i+1}}}(s)\,du.\end{aligned}$$

Expressing forward bond prices as

$$P(t_0,t,t+\delta) := \frac{P(t_0,t+\delta)}{P(t_0,t)} = \exp\left[-\int_t^{t+\delta} f(t_0,u)\,du\right],$$

and using Fubini's theorem to change the order of integration,

$$\ln P^{-1}(t,t+\delta) = \ln\left[\frac{P(t_0,t)}{P(t_0,t+\delta)}\right] + l(t_0,t) + \int_{t_0}^{t}\left[\underline{\sigma}(s,t)-\underline{\sigma}(s,t+\delta)\right]'\cdot d\underline{W}^{\mathcal{Q}^{t_{i+1}}}(s), \tag{18}$$

where $l(t_0,t)$ is defined by equation (16).

Considering that $g(t_0,t)<\infty$, as given by equation (17), and using, for instance, Arnold (1992, corollary 4.5.6), it follows that the stochastic integral contained in the last equality is normally distributed with mean zero and variance $g(t_0,t)$. ■

Prices for regular caplets and floorlets can be easily obtained by integrating the corresponding terminal payoffs with respect to the unrestricted density (15). For convenience, proposition 2 reminds such analytical solutions.

Proposition 2 *Under the Gaussian HJM specification (4), the time-*t_0 *price of a unit face value regular caplet (floorlet) on the nominal spot rate* $r_n(t_i,t_{i+1})$*, with a cap (floor) rate of* r_k*, and settled in arrears at time* t_{i+1} *is equal to*

$$\begin{aligned}REGULAR(\theta)_{t_0}\left[r_n(t_i,t_{i+1});r_k;t_{i+1}\right] &= \theta P(t_0,t_i)\,\Phi\left[-\theta d_0^{r_k}(t_0)+\theta\sqrt{g(t_0,t_i)}\right] \\ &\quad -\theta(1+\delta r_k)\,P(t_0,t_{i+1})\,\Phi\left[-\theta d_0^{r_k}(t_0)\right],\end{aligned} \tag{19}$$

with

$$d_0^{r}(t) = \frac{\ln\left[\frac{P(t,t_i,t_{i+1})}{(1+\delta r)^{-1}}\right]+\frac{g(t,t_i)}{2}}{\sqrt{g(t,t_i)}},$$

$\theta = 1$ *for a regular caplet,* $\theta = -1$ *for a regular floorlet, and where* Φ *represents the cumulative density function of the univariate standard normal distribution.*

[12] Hereafter, $\phi(X;\mu,\sigma) := (2\pi\sigma^2)^{-\frac{1}{2}}\exp\left[-\frac{1}{2}\frac{(X-\mu)^2}{\sigma^2}\right]$ represents the probability density function of a normally distributed univariate random variable X, with mean μ and standard deviation σ. $\Pr_{\mathcal{Q}^{t_{i+1}}}(\omega|\mathcal{F}_{t_0})$ denotes the probability of event ω, conditional on $\mathcal{F}_{t_0}$, and computed under the equivalent martingale measure $\mathcal{Q}^{t_{i+1}}$.

[13] $\|\cdot\|$ denotes the Euclidean norm in $\Re^n$.

Proof. See, for instance, Brace and Musiela (1994b, section 2.2). Alternatively, it is only necessary to compute an explicit solution for the following integral:

$$\begin{aligned} & REGULAR\left(\theta\right)_{t_0}\left[r_n\left(t_i,t_{i+1}\right);r_k;t_{i+1}\right] \\ = \; & P\left(t_0,t_{i+1}\right)\int_{-\infty}^{\infty}\left[\theta\exp\left(x\right)-\theta\left(1+\delta r_k\right)\right]^{+}\phi\left\{x;\ln\left[\frac{P\left(t_0,t_i\right)}{P\left(t_0,t_{i+1}\right)}\right]+l\left(t_0,t_i\right),\sqrt{g\left(t_0,t_i\right)}\right\}dx, \end{aligned} \tag{20}$$

using the fact that

$$l\left(t_0,t_i\right)=-\frac{1}{2}g\left(t_0,t_i\right). \tag{21}$$

■

Remark 6 *Equation (18) shows that function $g\left(t_0,t_i\right)$ is, as usual, the time-t_0 variance of the time-t_i log-price of a time-t_{i+1} maturity pure discount bond.*

3.3 Restricted densities for log-inverse pure discount bond prices

Next theorem contains the main theoretical result from this paper.

Theorem 1 *Under the Gaussian HJM specification (4), assuming that the deterministic volatility function $\underline{\sigma}\left(\cdot,T\right):\left[t_0,T\right]\rightarrow\Re^n$ is such that*

$$\frac{\partial g\left(t_0,t\right)}{\partial t}>0, \tag{22}$$

and using approximation (34), then:

$$\begin{aligned} & \Pr_{\mathcal{Q}^{t_{i+1}}}\left\{\ln P^{-1}\left(t,t+\delta\right)\in dx\wedge\sup_{t_0\leq s\leq t}\left[\theta\ln P^{-1}\left(s,s+\delta\right)\right]<\theta\ln\left(1+\delta r_H\right)\middle|\mathcal{F}_{t_0}\right\} \\ \approx \; & \left\{\phi\left[x;\ln P^{-1}\left(t_0,t_0+\delta\right)-\sqrt{h\left(t_0,t\right)g\left(t_0,t\right)},\sqrt{g\left(t_0,t\right)}\right]-\exp\left[2\sqrt{\frac{h\left(t_0,t\right)}{g\left(t_0,t\right)}}\ln\frac{P^{-1}\left(t_0,t_0+\delta\right)}{1+\delta r_H}\right]\right. \\ & \left.\phi\left[x;2\ln\left(1+\delta r_H\right)-\ln P^{-1}\left(t_0,t_0+\delta\right)-\sqrt{h\left(t_0,t\right)g\left(t_0,t\right)},\sqrt{g\left(t_0,t\right)}\right]\right\}dx, \end{aligned} \tag{23}$$

with

$$h\left(t_0,t\right):=\int_{t_0}^{t}\left[f\left(t_0,v+\delta\right)-f\left(t_0,v\right)+\frac{\partial l\left(t_0,v\right)}{\partial v}\right]^2\left[\frac{\partial g\left(t_0,v\right)}{\partial v}\right]^{-1}dv, \tag{24}$$

and where $\theta=1$ and $r_H=r_u(>r_n\left(t_0,t_0+\delta\right))$ for an up barrier or $\theta=-1$ and $r_H=r_l(<r_n\left(t_0,t_0+\delta\right))$ for a down barrier.

Proof. Following, for instance, Øksendal (1995, theorem 8.20), because $g\left(t_0,t_0\right)=0$ and using assumption (22), a time-change can be applied to equation (18):

$$\ln P^{-1}\left(t,t+\delta\right)=\ln\left[\frac{P\left(t_0,t\right)}{P\left(t_0,t+\delta\right)}\right]+l\left(t_0,t\right)+\tilde{W}^{\mathcal{Q}^{t_{i+1}}}_{g\left(t_0,t\right)}, \tag{25}$$

where

$$\tilde{W}^{\mathcal{Q}^{t_{i+1}}}_{g\left(t_0,t\right)}:=\int_{t_0}^{t}\left[\underline{\sigma}\left(s,t\right)-\underline{\sigma}\left(s,t+\delta\right)\right]'\cdot d\underline{W}^{\mathcal{Q}^{t_{i+1}}}\left(s\right) \tag{26}$$

is a one-dimensional Brownian motion also initialized at zero, under the same martingale measure $\mathcal{Q}^{t_{i+1}}$, but with respect to a deterministic time-change defined by $g\left(t_0,\cdot\right):\left[t_0,T\right]\rightarrow\Re^{+}$.

In order to remove the drift from the process (25), define an equivalent martingale measure $\mathcal{Q}^g$ such that

$$d\tilde{W}^{\mathcal{Q}^g}_{g\left(t_0,t\right)}=d\tilde{W}^{\mathcal{Q}^{t_{i+1}}}_{g\left(t_0,t\right)}+\eta\left(t_0,t\right)dg\left(t_0,t\right), \tag{27}$$

with

$$\eta(t_0,t) := \frac{\partial}{\partial g(t_0,t)} \left\{ \ln\left[\frac{P(t_0,t)}{P(t_0,t+\delta)}\right] + l(t_0,t) \right\}, \tag{28}$$

is also a time-changed one-dimensional Wiener increment. That is, define the change of measure

$$\left.\frac{dQ^{t_{i+1}}}{dQ^g}\right| \mathcal{F}_t = \exp\left\{ -\int_{t_0}^{t} [-\eta(t_0,v)]\, d\tilde{W}^{Q^g}_{g(t_0,v)} - \frac{1}{2}\int_{t_0}^{t} [-\eta(t_0,v)]^2\, dg(t_0,v) \right\}, \tag{29}$$

and assume that $E^{Q^g}_{t_0}\left\{\exp\left[\frac{1}{2}\int_{t_0}^{t}\eta(t_0,v)^2\, dg(t_0,v)\right]\right\} < \infty$. Integrating identity (27) between t_0 and t, and since $l(t_0,t_0)=0$, it follows that

$$\tilde{W}^{Q^g}_{g(t_0,t)} = \ln\left[\frac{P(t_0,t)}{P(t_0,t+\delta)}\right] + l(t_0,t) + \tilde{W}^{Q^{t_{i+1}}}_{g(t_0,t)} + \ln P(t_0,t_0+\delta).$$

Consequently equation (25) can be rewritten as a time-changed standard Brownian motion initialized at $[-\ln P(t_0,t_0+\delta)]$,

$$\ln P^{-1}(t,t+\delta) = \tilde{W}^{Q^g}_{g(t_0,t)} - \ln P(t_0,t_0+\delta), \tag{30}$$

and therefore

$$\begin{aligned} &\Pr_{Q^{t_{i+1}}}\left\{ \ln P^{-1}(t,t+\delta) \le x \wedge \sup_{t_0\le s\le t}\left[\theta \ln P^{-1}(s,s+\delta)\right] < \theta\ln(1+\delta r_H) \right| \left.\mathcal{F}_{t_0}\right\} \quad (31)\\ = &\Pr_{Q^{t_{i+1}}}\left\{ \tilde{W}^{Q^g}_{g(t_0,t)} \le x + \ln P(t_0,t_0+\delta) \wedge \sup_{t_0\le s\le t}\left[\theta \tilde{W}^{Q^g}_{g(t_0,s)}\right] < \theta \ln\left[\frac{1+\delta r_H}{P^{-1}(t_0,t_0+\delta)}\right] \right| \left.\mathcal{F}_{t_0}\right\}. \end{aligned}$$

It would be simple to compute the right-hand-side of the previous equality if the probability were defined under measure Q^g. Since this is not the case, it will be necessary to use the Radon-Nikodym derivative (29) -see, for instance Duffie (1992, equation C.3)- and to express it as a function of the random variable $\tilde{W}^{Q^g}_{g(t_0,t)}$. For that purpose, notice that the Itô integral contained in the Radon-Nikodym derivative (29) possess a univariate normal distribution with zero mean and variance $h(t_0,t)$, that is[14]

$$\int_{t_0}^{t} [-\eta(t_0,v)]\, d\tilde{W}^{Q^g}_{g(t_0,v)} \backsim N^1(0, h(t_0,t)), \tag{32}$$

where the symbol $\backsim$ denotes equality in distribution, and

$$h(t_0,t) := \int_{t_0}^{t} \eta(t_0,v)^2\, dg(t_0,v)$$

can be expressed as in equation (24), using definition (28). Consequently, and since $\tilde{W}^{Q^g}_{g(t_0,t)} \backsim N^1(0, g(t_0,t))$, then

$$\int_{t_0}^{t} [-\eta(t_0,v)]\, d\tilde{W}^{Q^g}_{g(t_0,v)} \backsim \sqrt{\frac{h(t_0,t)}{g(t_0,t)}}\tilde{W}^{Q^g}_{g(t_0,t)}, \tag{33}$$

which suggests the following *proportionality approximation* for the Radon-Nikodym derivative (29).

Assumption 3 The Radon-Nikodym derivative (29) will be approximated by

$$\left.\frac{dQ^{t_{i+1}}}{dQ^g}\right| \mathcal{F}_t \approx \exp\left[-\sqrt{\frac{h(t_0,t)}{g(t_0,t)}}\tilde{W}^{Q^g}_{g(t_0,t)} - \frac{1}{2}h(t_0,t) \right]. \tag{34}$$

[14] The notation $X \backsim N^1(\mu,\sigma^2)$ is intended to mean that the one-dimensional random variable X is normally distributed, with mean μ, and variance σ^2.

Remark 7 *Approximation (34) is in the spirit of the proportionality assumption used by El Karoui and Rochet (1989, page 22) or of the rank 1 approximation suggested by Brace and Musiela (1994a, equation 6.1), which have both been shown to be accurate in the context of European swaption pricing -see, for instance, Brace and Musiela (1994a, table 7.5) or Pang (1996, table 6). Nevertheless, since the above equality is only exact in distribution, all the approximate pricing solutions that will be derived will be subject to a Monte Carlo study, in order to investigate their accuracy.*

Finally, in order to compute probability (31) explicitly, the following result must be used -see, for instance Harrison (1985, page 9), Borodin and Salminen (1996, page 104) or Musiela and Rutkowski (1998, corollary B.3.1 and corollary B.3.3):

Lemma 1 *If $W^{\mathcal{P}}(t)$ is a $\mathcal{P}$-measured standard one-dimensional Brownian motion, such that $W^{\mathcal{P}}(t_0) = 0$, then*

$$\Pr_{\mathcal{P}}\left\{ W^{\mathcal{P}}(t) \in dx \wedge \sup_{t_0 \le s \le t} \left[\theta W^{\mathcal{P}}(s)\right] < y \middle| \mathcal{F}_{t_0} \right\} = \left[\phi\left(x; 0, \sqrt{t-t_0}\right) - \phi\left(x; 2y, \sqrt{t-t_0}\right)\right] dx, \tag{35}$$

where $y \ge 0$ and $\theta = \{-1, 1\}$.

Remark 8 *Using definition (6), it is easy to check that the barrier level $\ln\left[\frac{1+\delta r_H}{P^{-1}(t_0,t_0+\delta)}\right]$ is positive for an up barrier -$r_H = r_u (> r_n(t_0, t_0+\delta))$- and negative for a down barrier -$r_H = r_l (< r_n(t_0, t_0+\delta))$.*

Using lemma 1 and the convergence in law result (34), equation (31) becomes

$$\begin{aligned} & \Pr_{\mathcal{Q}^{t_{i+1}}}\left\{ \ln P^{-1}(t, t+\delta) \le x \wedge \sup_{t_0 \le s \le t}\left[\theta \ln P^{-1}(s, s+\delta)\right] < \theta \ln(1+\delta r_H) \middle| \mathcal{F}_{t_0} \right\} \\ = & \int_{-\infty}^{x+\ln P(t_0,t_0+\delta)} \exp\left[-\sqrt{\frac{h(t_0,t)}{g(t_0,t)}}z - \frac{1}{2}h(t_0,t)\right] \\ & \left\{\phi\left[z; 0, \sqrt{g(t_0,t)}\right] - \phi\left[z; 2\ln\frac{1+\delta r_H}{P^{-1}(t_0,t_0+\delta)}, \sqrt{g(t_0,t)}\right]\right\} dz. \end{aligned} \tag{36}$$

Differentiating the above integral with respect to x,

$$\begin{aligned} & \Pr_{\mathcal{Q}^{t_{i+1}}}\left\{ \ln P^{-1}(t, t+\delta) \in dx \wedge \sup_{t_0 \le s \le t}\left[\theta \ln P^{-1}(s, s+\delta)\right] < \theta \ln(1+\delta r_H) \middle| \mathcal{F}_{t_0} \right\} \\ = & \exp\left\{-\frac{1}{2}h(t_0,t) - \sqrt{\frac{h(t_0,t)}{g(t_0,t)}}\left[x + \ln P(t_0,t_0+\delta)\right]\right\} \\ & \left\{\phi\left[x + \ln P(t_0,t_0+\delta); 0, \sqrt{g(t_0,t)}\right] - \phi\left[x + \ln P(t_0,t_0+\delta); 2\ln\frac{1+\delta r_H}{P^{-1}(t_0,t_0+\delta)}, \sqrt{g(t_0,t)}\right]\right\} dx, \end{aligned}$$

and using the definition $\phi(X; \mu, \sigma) := \left(2\pi\sigma^2\right)^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\frac{(X-\mu)^2}{\sigma^2}\right]$, equation (23) follows. ■

Corollary 1 *Under the Gaussian HJM specification (4), and subject to assumptions (22) and (34),*

$$\begin{aligned} & \Pr_{\mathcal{Q}^{t_{i+1}}}\left\{ \ln P^{-1}(t, t+\delta) \in dx \wedge \sup_{t_0 \le s \le t}\left[\theta \ln P^{-1}(s, s+\delta)\right] \ge \theta \ln(1+\delta r_H) \middle| \mathcal{F}_{t_0} \right\} \\ \approx & \; dx \begin{cases} \phi\left[x; \ln\frac{P(t_0,t)}{P(t_0,t+\delta)} + l(t_0,t), \sqrt{g(t_0,t)}\right] \\ -\phi\left[x; \ln P^{-1}(t_0,t_0+\delta) - \sqrt{h(t_0,t)\,g(t_0,t)}, \sqrt{g(t_0,t)}\right] \\ +\exp\left[2\sqrt{\frac{h(t_0,t)}{g(t_0,t)}}\ln\frac{P^{-1}(t_0,t_0+\delta)}{1+\delta r_H}\right] \\ \phi\left[x; \ln(1+\delta r_H)^2 - \ln P^{-1}(t_0,t_0+\delta) - \sqrt{h(t_0,t)\,g(t_0,t)}, \sqrt{g(t_0,t)}\right] & \Longleftarrow \theta x < \theta\ln(1+\delta r_H) \\ \\ \phi\left[x; \ln\frac{P(t_0,t)}{P(t_0,t+\delta)} + l(t_0,t), \sqrt{g(t_0,t)}\right] & \Longleftarrow \theta x \ge \theta\ln(1+\delta r_H) \end{cases} \end{aligned} \tag{37}$$

where $\theta = 1$ *and* $r_H = r_u (> r_n (t_0, t_0 + \delta))$ *for an up barrier or* $\theta = -1$ *and* $r_H = r_l (< r_n (t_0, t_0 + \delta))$ *for a down barrier.*

Proof. Equation (37) arises from theorem 1 and proposition 1, through the following obvious identity:

$$\begin{aligned}
& \Pr_{Q^{t_{i+1}}} \left\{ \ln P^{-1}(t, t+\delta) \in dx \wedge \sup_{t_0 \le s \le t} \left[\theta \ln P^{-1}(s, s+\delta) \right] \ge \theta \ln (1 + \delta r_H) \middle| \mathcal{F}_{t_0} \right\} \\
= \; & \Pr_{Q^{t_{i+1}}} \left[\ln P^{-1}(t, t+\delta) \in dx \middle| \mathcal{F}_{t_0} \right] \\
& - \Pr_{Q^{t_{i+1}}} \left\{ \ln P^{-1}(t, t+\delta) \in dx \wedge \sup_{t_0 \le s \le t} \left[\theta \ln P^{-1}(s, s+\delta) \right] < \theta \ln (1 + \delta r_H) \middle| \mathcal{F}_{t_0} \right\}.
\end{aligned}$$

■

Equipped with theorem 1 and corollary 1, it is now possible to price any type of barrier cap or floor.

3.4 Pricing formulae

In order to price single barrier caps and floors it is only necessary to compute the expectations contained in equations (11) and (12), using the probability densities offered by theorem 1 or corollary 1.[15]

Proposition 3 *Under the Gaussian HJM specification (4), and subject to condition (22) and approximation (34),*

$$\begin{aligned}
& IN(\theta)_{t_0} \left[r_n(t_i, t_{i+1}); r_k; r_H; R = 0; t_{i+1} \right] \qquad (38) \\
= \; & REGULAR(\theta)_{t_0} \left[r_n(t_i, t_{i+1}); r_k; t_{i+1} \right] \\
& + 1_{\{\theta r_H > \theta r_k\}} (1 + \delta r_k) P(t_0, t_{i+1}) \left\{ \Phi\left[d_1^{r_H}(t_0) \right] - \Phi\left[d_1^{r_k}(t_0) \right] \right\} \\
& - 1_{\{\theta r_H > \theta r_k\}} P(t_0, t_0 + \delta, t_{i+1}) \exp\left[\frac{1}{2} g(t_0, t_i) - \sqrt{h(t_0, t_i) g(t_0, t_i)} \right] \\
& \left\{ \Phi\left[d_1^{r_H}(t_0) - \sqrt{g(t_0, t_i)} \right] - \Phi\left[d_1^{r_k}(t_0) - \sqrt{g(t_0, t_i)} \right] \right\} \\
& + 1_{\{\theta r_H > \theta r_k\}} \exp\left\{ 2 \sqrt{\frac{h(t_0, t_i)}{g(t_0, t_i)}} \ln\left[\frac{P^{-1}(t_0, t_0 + \delta)}{1 + \delta r_H} \right] \right\} \left\{ P(t_0, t_{i+1}) P(t_0, t_0 + \delta) (1 + \delta r_H)^2 \right. \\
& \exp\left[\frac{1}{2} g(t_0, t_i) - \sqrt{h(t_0, t_i) g(t_0, t_i)} \right] \left[\Phi\left(d_2^{r_H}(t_0) - \sqrt{g(t_0, t_i)} \right) - \Phi\left(d_2^{r_k}(t_0) - \sqrt{g(t_0, t_i)} \right) \right] \\
& \left. - (1 + \delta r_k) P(t_0, t_{i+1}) \left[\Phi\left(d_2^{r_H}(t_0) \right) - \Phi\left(d_2^{r_k}(t_0) \right) \right] \right\},
\end{aligned}$$

with

$$d_1^r(t) = \frac{\ln\left[\frac{P(t, t+\delta)}{(1+\delta r)^{-1}} \right] + \sqrt{h(t, t_i) g(t, t_i)}}{\sqrt{g(t, t_i)}},$$

$$d_2^r(t) = \frac{\ln\left[\frac{P^{-1}(t, t+\delta)}{(1+\delta r)^{-1} (1+\delta r_H)^2} \right] + \sqrt{h(t, t_i) g(t, t_i)}}{\sqrt{g(t, t_i)}},$$

and where $r_H = r_u (> r_n (t_0, t_0 + \delta))$ *and* $\theta = 1$ *for an up-and-in caplet, or* $r_H = r_l (< r_n (t_0, t_0 + \delta))$ *and* $\theta = -1$ *for a down-and-in floorlet.*

Proof. See appendix A. ■

Remark 9 *As expected, when* $r_u < r_k$ $(r_l > r_k)$ *the barrier price collapses into a regular caplet (floorlet).*

[15] Next formulae use an indicator function, defined as:

$$1_{\{\omega \in \Omega\}} = \left\{ \begin{array}{l} 1 \Longleftarrow \omega \in \Omega \\ 0 \Longleftarrow \omega \notin \Omega \end{array} \right. .$$

Remark 10 *The fair price of an up-and-out caplet (down-and-out floorlet) can be easily obtained by taking the difference between equations (19) and (38), with $r_H = r_u$ and $\theta = 1$ ($r_H = r_l$ and $\theta = -1$). This result follows from the well known parity relation between knock-in and knock-out options without rebate.*

Proposition 4 *Under the Gaussian HJM specification (4), and subject to condition (22) and approximation (34),*

$$
\begin{aligned}
& OUT(\theta)_{t_0}\left[r_n(t_i,t_{i+1});r_k;r_H;R=0;t_{i+1}\right] \qquad (39) \\
= \; & \theta P(t_0,t_0+\delta,t_{i+1})\exp\left[\frac{1}{2}g(t_0,t_i)-\sqrt{h(t_0,t_i)\,g(t_0,t_i)}\right]\Phi\left[\theta\sqrt{g(t_0,t_i)}-\theta d_1^{\theta\max(\theta r_H,\theta r_k)}(t_0)\right] \\
& -\theta(1+\delta r_k)P(t_0,t_{i+1})\Phi\left[-\theta d_1^{\theta\max(\theta r_H,\theta r_k)}(t_0)\right] \\
& -\theta\exp\left\{2\sqrt{\frac{h(t_0,t)}{g(t_0,t)}}\ln\left[\frac{P^{-1}(t_0,t_0+\delta)}{1+\delta r_H}\right]\right\}\left\{P(t_0,t_{i+1})P(t_0,t_0+\delta)(1+\delta r_H)^2\right. \\
& \exp\left[\frac{1}{2}g(t_0,t_i)-\sqrt{h(t_0,t_i)\,g(t_0,t_i)}\right]\Phi\left[\theta\sqrt{g(t_0,t_i)}-\theta d_2^{\theta\max(\theta r_H,\theta r_k)}(t_0)\right] \\
& \left.-(1+\delta r_k)P(t_0,t_{i+1})\Phi\left[-\theta d_2^{\theta\max(\theta r_H,\theta r_k)}(t_0)\right]\right\},
\end{aligned}
$$

where $r_H = r_l (< r_n(t_0,t_0+\delta))$ and $\theta = 1$ for an down-and-out caplet, or $r_H = r_u (> r_n(t_0,t_0+\delta))$ and $\theta = -1$ for a up-and-out floorlet.

Proof. See appendix B. ■

Remark 11 *The fair price of an down-and-in caplet (up-and-in floorlet) can also be easily obtained by taking the difference between equations (19) and (39), with $r_H = r_l$ and $\theta = 1$ ($r_H = r_u$ and $\theta = -1$).*

3.5 Examples

In order to test the efficiency and accuracy of the proposed pricing approximations, two examples will be considered. Although propositions 3 and 4 are applicable to all deterministic specifications of the volatility function (even time-dependent ones), next examples consider a simple Gauss-Markov time-homogeneous HJM model, which can be defined through the following proposition.

Proposition 5 *If the short-term interest rate is Markovian and the volatility function $\underline{\sigma}(\cdot,T):[t_0,T]\to\Re^n$ is time-homogeneous, then the volatility function must be restricted to the following analytical specification:*

$$\underline{\sigma}(t,T)' := \underline{\sigma}(T-t)' = \underline{G}'\cdot a^{-1}\cdot\left[I_n - e^{a(T-t)}\right], \qquad (40)$$

where, $I_n \in \Re^{n\times n}$ represents an identity matrix, while $\underline{G}\in\Re^n$ and $a\in\Re^{n\times n}$ contain model' time-independent parameters.

Proof. Proposition 5 follows, for instance, from Musiela and Rutkowski (1998, proposition 13.3.2). ■

Remark 12 *Appendix C summarizes the simpler explicit solutions that can be obtained for functions $g(t,t_i)$ and $h(t,t_i)$, under proposition 5.*

The Gauss-Markov time-homogeneous HJM model specification adopted in the following examples is defined by a flat spot yield curve (continuously compounded spot interest rates are equal to 6%, for all maturities), and by the following volatility specification:

$$\underline{G} = \left[\begin{array}{ccc} 0.004243 & 0.005657 & 0.007071 \end{array}\right]', a = diag\{-0.1,-0.15,-0.2\}.$$

Such three-factor HJM model specification was borrowed from the Gaussian nested affine formulation contained in Nunes et al. (1999, Table 5).

Table 1 prices a three-year (unit face value) up-and-in cap with a cap rate of 6.045%, a barrier rate of 7% and quarterly compounding. Exact up-and-in caplet prices were estimated through standard Monte Carlo simulation, using the usual Euler discretization of equation (4) with 520 time steps per year, independent normal variates generated through the Box-Muller algorithm, and $200,000$ simulations. It is assumed that the barrier is monitored 260 times per year (i.e., daily monitoring). Besides the Monte Carlo price estimate, the percentage of its standard error on the mean price is also shown. Approximate barrier caplet prices were computed from proposition 3, while their regular counterparts were obtained from proposition 2. The corresponding percentage (pricing) errors were computed with respect to the (exact) Monte Carlo price estimate. Throughout this paper, the CPU time is always shown in seconds (except if stated otherwise), and all computations are made running Pascal programs on a Pentium 400Mhz with 64MB of RAM memory.

The proposed approximate solution -proposition 3- is very fast to implement (11.7 seconds) and relatively accurate (pricing errors of about 1%). Notice also that, because of the high level chosen for the barrier rate (7%), the regular caplet prices are significantly different from their up-and-in counterparts (the up-and-in cap is about 20% cheaper). Although the pricing errors are of about three or four standard errors, one should keep in mind that two error sources are affecting the results. One is the use of approximation (34). The second error source is the fact that proposition 3 assumes continuous monitoring of the barrier, while the Monte Carlo price estimate (that is the proxy for the exact price) considers that the barrier is discretely monitored. And, as argued and shown, for instance, by Broadie, Glasserman and Kou (1997), "...there can be substantial price differences between discrete and continuous barrier options, even under daily monitoring of the barrier". Therefore, it would be extremely important if propositions 3 and 4 could be adapted for discrete monitoring using a continuity correction similar to the one provided by Broadie et al. (1997, theorem 1.1) and Broadie, Glasserman and Kou (1999, theorem 1). Such development is left for further research.

Table 2 considers a two-year (unit face value) down-and-out cap with a cap rate of 6.045%, a barrier rate of 5% and quarterly compounding. Approximate barrier caplet prices were computed from proposition 4, while the Monte Carlo price estimates still assume 520 time-steps per year, daily monitoring, and are based on $200,000$ simulations. Again, such approximate solution is very fast, although the pricing errors increase with the option maturity.

4 Partial-barrier caps and floors

4.1 Definitions

For the single-barrier caps and floors studied in section 3, the barrier' monitoring periods are not independent between different component barrier caplets or floorlets. That is, if the barrier is knocked-in for some component barrier caplet or floorlet, all the subsequent caplets or floorlets become regular. Similarly, if the barrier is knocked-out for some component barrier caplet or floorlet, all the subsequent caplets or floorlets become worthless. In summary, once the barrier is touched, all the path-dependency is resolved during the remaining life of the cap or floor.

One alternative configuration of such barrier caps and floors is to make the barrier' monitoring period of each component caplet or floorlet independent from the others. This can be accomplished if the barrier' monitoring period for the component caplet or floorlet on the nominal spot rate $r_n(t_i, t_{i+1})$ is no longer the time interval $]t_0, t_i]$ but rather the previous compounding period $[t_{i-1}, t_i]$. Hence, such component caplet or floorlet can expire worthless even if the barrier has been knocked-in before time t_{i-1}, and it is also possible that the terminal payoff is positive even if the barrier has been knocked-out before time t_{i-1}. Borrowing the terminology from Heynen and Kat (1994), these barrier caplets and floorlets with barrier' monitoring periods restricted to a subset of the option's life will be called *partial-barrier caplets and floorlets.*

Definition 3 *The time-t_{i+1} price of a unit face value partial up-and-in caplet (partial down-and-in floorlet) on the nominal spot rate $r_n(t_i, t_{i+1})$, with a cap (floor) rate of r_k, subject to a barrier rate of r_u (r_l), with a rebate rate R, and settled in arrears at time t_{i+1} is equal to*

$$IN(\theta)^P_{t_{i+1}}[r_n(t_i,t_{i+1}); r_k; r_H; R; t_{i+1}] := \begin{cases} [\theta r_n(t_i,t_{i+1}) - \theta r_k]^+ \delta \Longleftarrow \exists s \in [t_{i-1}, t_i] : \theta r_n(s, s+\delta) \geq \theta r_H \\ R\delta \Longleftarrow \theta r_n(s, s+\delta) < \theta r_H, \forall s \in [t_{i-1}, t_i] \end{cases}, \tag{41}$$

where $r_H = r_u (> r_n (t_0, t_0 + \delta))$ and $\theta = 1$ for a partial up-and-in caplet, or $r_H = r_l (< r_n (t_0, t_0 + \delta))$ and $\theta = -1$ for a partial down-and-in floorlet.

Definition 4 *The time-t_{i+1} price of a unit face value partial down-and-out caplet (partial up-and-out floorlet) on the nominal spot rate $r_n (t_i, t_{i+1})$, with a cap (floor) rate of r_k, subject to a barrier rate of r_l (r_u), with a rebate rate R, and settled in arrears at time t_{i+1} is equal to*

$$OUT(\theta)^P_{t_{i+1}} [r_n (t_i, t_{i+1}); r_k; r_H; R; t_{i+1}] := \begin{cases} [\theta r_n (t_i, t_{i+1}) - \theta r_k]^+ \delta \Longleftarrow \theta r_n (s, s+\delta) > \theta r_H, \forall s \in [t_{i-1}, t_i] \\ R\delta \Longleftarrow \exists s \in [t_{i-1}, t_i] : \theta r_n (s, s+\delta) \leq \theta r_H \end{cases}, \tag{42}$$

where $r_H = r_l (< r_n (t_0, t_0 + \delta))$ and $\theta = 1$ for a partial down-and-out caplet, or $r_H = r_u (> r_n (t_0, t_0 + \delta))$ and $\theta = -1$ for a partial up-and-out floorlet.

In order to value the above mentioned partial-barrier options, two approaches can be used. One possibility is to compute the discounted expectation, under measure $\mathcal{Q}^{t_{i+1}}$, of the terminal payoff. For this purpose, however, the restricted density offered by theorem 1 would have to be modified in order for the monitoring period to be restricted from $[t_0, t_i]$ to $[t_{i-1}, t_i]$. A simpler approach results from noticing that, at time t_{i-1} (beginning of the monitoring period), a partial-barrier caplet or floorlet, with zero rebate, on the nominal spot rate $r_n (t_i, t_{i+1})$ is exactly equivalent to a single-barrier caplet or floorlet, that is can be valued through propositions 3 or 4, when t_0 is replaced by t_{i-1}. Therefore, and following Harrison and Pliska (1981),

$$\begin{aligned} & IN(\theta)^P_{t_0} [r_n (t_i, t_{i+1}); r_k; r_H; R = 0; t_{i+1}] \\ = \; & P(t_0, t_{i-1}) E^{\mathcal{Q}^{t_{i-1}}}_{t_0} \left\{ IN(\theta)_{t_{i-1}} [r_n (t_i, t_{i+1}); r_k; r_H; R = 0; t_{i+1}] \middle| \theta r_n (t_{i-1}, t_i) < \theta r_H \right\} \\ & + P(t_0, t_{i-1}) E^{\mathcal{Q}^{t_{i-1}}}_{t_0} \left\{ REGULAR(\theta)_{t_{i-1}} [r_n (t_i, t_{i+1}); r_k; t_{i+1}] \middle| \theta r_n (t_{i-1}, t_i) \geq \theta r_H \right\}, \end{aligned} \tag{43}$$

and

$$\begin{aligned} & OUT(\theta)^P_{t_0} [r_n (t_i, t_{i+1}); r_k; r_H; R = 0; t_{i+1}] \\ = \; & P(t_0, t_{i-1}) E^{\mathcal{Q}^{t_{i-1}}}_{t_0} \left\{ OUT(\theta)_{t_{i-1}} [r_n (t_i, t_{i+1}); r_k; r_H; R = 0; t_{i+1}] \middle| \theta r_n (t_{i-1}, t_i) > \theta r_H \right\}, \end{aligned} \tag{44}$$

where $\mathcal{Q}^{t_{i-1}}$ is the t_{i-1}-forward martingale measure obtained when the t_{i-1}-maturity pure discount bond is taken as the numeraire, which is assumed to exist and to be defined through the following *Radon-Nikodym derivative*:

$$\left. \frac{d\mathcal{Q}^{t_{i-1}}}{d\mathcal{Q}} \right| \mathcal{F}_t := \frac{P(t, t_{i-1})}{P(t_0, t_{i-1})} \frac{\beta(t_0)}{\beta(t)}. \tag{45}$$

The option prices $REGULAR(\theta)_{t_{i-1}}[\cdot]$, $IN(\theta)_{t_{i-1}}[\cdot]$, and $OUT(\theta)_{t_{i-1}}[\cdot]$ are simply given by propositions 2, 3, and 4, respectively, when t_0 is replaced by t_{i-1}, and only include two types of $\mathcal{F}_{t_{i-1}}$-measurable random variables, namely: time-t_{i-1} zero-coupon bond prices, and function $h(t_{i-1}, t_i)$, as defined by equation (24). Next subsection derives the density function of time-t_{i-1} discount factors, under measure $\mathcal{Q}^{t_{i-1}}$. Concerning the stochastic function $h(t_{i-1}, t_i)$, the following simplifying assumption will be used.

Assumption 4 The random variable $h(t_{i-1}, t_i)$ will be approximated by the following deterministic function:

$$h(t_{i-1}, t_i) \approx \hat{h}(t_{i-1}, t_i) := \int_{t_{i-1}}^{t_i} \left[f(t_0, v + \delta) - f(t_0, v) + \frac{\partial l(t_{i-1}, v)}{\partial v} \right]^2 \left[\frac{\partial g(t_{i-1}, v)}{\partial v} \right]^{-1} dv. \tag{46}$$

Remark 13 *The previous simplification is in the spirit of the assumption made by Brace, Gatarek and Musiela (1997, equation 3.17) in order to obtain an approximation for European swaption prices under the lognormal Libor Market Model. In our case, replacing $[f(t_{i-1}, v + \delta) - f(t_{i-1}, v)]$ by $[f(t_0, v + \delta) - f(t_0, v)]$ is tautological to assume that the slope of the forward yield curve, in δ-years intervals, is preserved between times t_0 and t_{i-1}. Such assumption is not too severe since δ is typically small (e.g. 0.25 years).*

4.2 Time-t_{i-1} discount factors under measure $Q^{t_{i-1}}$

Next proposition provides the probability law needed to compute explicitly the expectations contained in equations (43) and (44).

Proposition 6 *Under the equivalent martingale measure $Q^{t_{i-1}}$, the time-t_{i-1} price of a pure discount bond with maturity $T(\geq t_{i-1})$ is equal in distribution to:*

$$P(t_{i-1},T) \backsim P(t_0,t_{i-1},T)\exp\left[-\frac{1}{2}\varphi(t_0,t_{i-1},T)+\sqrt{\varphi(t_0,t_{i-1},T)}z\right], \tag{47}$$

with $z \backsim N^1(0,1)$ and where

$$\varphi(t_0,t_{i-1},T) := \int_{t_0}^{t_{i-1}} \left\|\underline{\sigma}(s,T)-\underline{\sigma}(s,t_{i-1})\right\|^2 ds. \tag{48}$$

Proof. From equation (2),

$$\begin{aligned}\ln\left[\frac{P(t,T)}{P(t,t_{i-1})}\right] &= \ln P(t_0,t_{i-1},T)-\frac{1}{2}\int_{t_0}^{t}\left[\left\|\underline{\sigma}(s,T)\right\|^2-\left\|\underline{\sigma}(s,t_{i-1})\right\|^2\right]ds \\ &\quad +\int_{t_0}^{t}\left[\underline{\sigma}(s,T)-\underline{\sigma}(s,t_{i-1})\right]'\cdot d\underline{W}^{Q}(s).\end{aligned} \tag{49}$$

Moreover, applying the same steps as in deriving identity (13), it is easy to show that assuming the existence of measure $Q^{t_{i-1}}$ implies that

$$d\underline{W}^{Q^{t_{i-1}}}(t) = d\underline{W}^{Q}(t)-\underline{\sigma}(t,t_{i-1})\,dt, \tag{50}$$

is a n-dimensional $Q^{t_{i-1}}$-measured vector of Brownian motion increments.

Combining equations (49) and (50), while making $t=t_{i-1}$,

$$\begin{aligned}P(t_{i-1},T) &= P(t_0,t_{i-1},T)\exp\left\{-\frac{1}{2}\int_{t_0}^{t_{i-1}}\left[\left\|\underline{\sigma}(s,T)\right\|^2-\left\|\underline{\sigma}(s,t_{i-1})\right\|^2\right]ds\right. \\ &\quad \left.+\int_{t_0}^{t_{i-1}}\left[\underline{\sigma}(s,T)-\underline{\sigma}(s,t_{i-1})\right]'\cdot\underline{\sigma}(s,t_{i-1})\,ds+\int_{t_0}^{t_{i-1}}\left[\underline{\sigma}(s,T)-\underline{\sigma}(s,t_{i-1})\right]'\cdot d\underline{W}^{Q^{t_{i-1}}}(s)\right\}.\end{aligned}$$

The first two integrals, inside the exponential, can be shown to be equal to $-\frac{1}{2}\varphi(t_0,t_{i-1},T)$. Finally, Arnold (1992, corollary 4.5.6) implies that

$$\int_{t_0}^{t_{i-1}}\left[\underline{\sigma}(s,T)-\underline{\sigma}(s,t_{i-1})\right]'\cdot d\underline{W}^{Q^{t_{i-1}}}(s) \backsim N^1\left(0,\varphi(t_0,t_{i-1},T)\right).$$

■

Remark 14 *Applying result (47) to different maturities T, as shall be done when solving equations (43) and (44), is similar to use the proportionality assumption (34).*

4.3 Pricing formulae

Equipped with proposition 6 and assumption (46), it is now possible to adjust propositions 3 and 4 for the case where the barrier' monitoring period is different for each component caplet or floorlet. Because equations (43) and (44) imply the integration of a univariate normal probability density function over combinations of univariate normal distribution functions, next pricing solutions are expected to involve the bivariate normal distribution function. In what follows, $M(a;b;\rho)$ represents the cumulative probability, in a standardized bivariate normal distribution, that the first variable is less than a and the second variable is less than b, when the coefficient of correlation between the variables is ρ.

Proposition 7 *Under the Gaussian HJM specification (4), and subject to condition (22) and approximations (34) and (46),*

$$
\begin{aligned}
& IN(\theta)_{t_0}^{P}\left[r_n(t_i,t_{i+1});r_k;r_H;R=0;t_{i+1}\right] \qquad (51)\\
= \quad & \theta P(t_0,t_i)\,\Phi\left\{\theta\sqrt{g(t_{i-1},t_i)+\left[\sqrt{\varphi(t_0,t_{i-1},t_{i+1})}-\sqrt{\varphi(t_0,t_{i-1},t_i)}\right]^2}-\theta p_0^{r_k}(t_0)\right\}\\
& -\theta(1+\delta r_k)\,P(t_0,t_{i+1})\,\Phi\left[-\theta p_0^{r_k}(t_0)\right]\\
& +1_{\{\theta r_H>\theta r_k\}}(1+\delta r_k)\,P(t_0,t_{i+1})\left\{M\left[-z^*+\sqrt{\varphi(t_0,t_{i-1},t_{i+1})};p_1^{r_H}(t_0);\theta\rho\right]\right.\\
& \left.-M\left[-z^*+\sqrt{\varphi(t_0,t_{i-1},t_{i+1})};p_1^{r_k}(t_0);\theta\rho\right]\right\}-1_{\{\theta r_H>\theta r_k\}}P(t_0,t_{i-1})\,P(t_0,t_i,t_{i+1})\\
& \exp\left[\frac{1}{2}g(t_{i-1},t_i)-\sqrt{\hat{h}(t_{i-1},t_i)\,g(t_{i-1},t_i)}+\varphi(t_0,t_{i-1},t_i)-\sqrt{\varphi(t_0,t_{i-1},t_i)\,\varphi(t_0,t_{i-1},t_{i+1})}\right]\\
& \left\{M\left[-z^*+\sqrt{\varphi(t_0,t_{i-1},t_{i+1})}-\sqrt{\varphi(t_0,t_{i-1},t_i)};p_1^{r_H}(t_0)-\sqrt{g(t_{i-1},t_i)+\varphi(t_0,t_{i-1},t_i)};\theta\rho\right]\right.\\
& \left.-M\left[-z^*+\sqrt{\varphi(t_0,t_{i-1},t_{i+1})}-\sqrt{\varphi(t_0,t_{i-1},t_i)};p_1^{r_k}(t_0)-\sqrt{g(t_{i-1},t_i)+\varphi(t_0,t_{i-1},t_i)};\theta\rho\right]\right\}\\
& +1_{\{\theta r_H>\theta r_k\}}P(t_0,t_{i+1})\,P(t_0,t_{i-1},t_i)\,(1+\delta r_H)^2\exp\left\{2\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\ln\left[\frac{P^{-1}(t_0,t_{i-1},t_i)}{1+\delta r_H}\right]\right.\\
& +\frac{g(t_{i-1},t_i)}{2}-\sqrt{\hat{h}(t_{i-1},t_i)\,g(t_{i-1},t_i)}+\left[1-2\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\right]\sqrt{\varphi(t_0,t_{i-1},t_i)}\left[\sqrt{\varphi(t_0,t_{i-1},t_{i+1})}\right.\\
& \left.\left.-\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\sqrt{\varphi(t_0,t_{i-1},t_i)}\right]\right\}\left\{M\left[-z^*+\sqrt{\varphi(t_0,t_{i-1},t_{i+1})}+\left(1-2\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\right)\right.\right.\\
& \left.\sqrt{\varphi(t_0,t_{i-1},t_i)};p_2^{r_H}(t_0)-\sqrt{g(t_{i-1},t_i)+\varphi(t_0,t_{i-1},t_i)};-\theta\rho\right]-M\left[-z^*+\sqrt{\varphi(t_0,t_{i-1},t_{i+1})}\right.\\
& \left.\left.+\left(1-2\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\right)\sqrt{\varphi(t_0,t_{i-1},t_i)};p_2^{r_k}(t_0)-\sqrt{g(t_{i-1},t_i)+\varphi(t_0,t_{i-1},t_i)};-\theta\rho\right]\right\}\\
& -1_{\{\theta r_H>\theta r_k\}}(1+\delta r_k)\,P(t_0,t_{i+1})\exp\left\{2\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\ln\left[\frac{P^{-1}(t_0,t_{i-1},t_i)}{1+\delta r_H}\right]+\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\right.\\
& \left.\sqrt{\varphi(t_0,t_{i-1},t_i)}\left[\left(1+2\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\right)\sqrt{\varphi(t_0,t_{i-1},t_i)}-2\sqrt{\varphi(t_0,t_{i-1},t_{i+1})}\right]\right\}\\
& \left\{M\left[-z^*+\sqrt{\varphi(t_0,t_{i-1},t_{i+1})}-2\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\sqrt{\varphi(t_0,t_{i-1},t_i)};p_2^{r_H}(t_0);-\theta\rho\right]\right.\\
& \left.-M\left[-z^*+\sqrt{\varphi(t_0,t_{i-1},t_{i+1})}-2\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\sqrt{\varphi(t_0,t_{i-1},t_i)};p_2^{r_k}(t_0);-\theta\rho\right]\right\},
\end{aligned}
$$

with

$$
z^*=\frac{\ln\left[\frac{P^{-1}(t_0,t_{i-1},t_i)}{1+\delta r_H}\right]+\frac{1}{2}\varphi(t,t_{i-1},t_i)}{\sqrt{\varphi(t,t_{i-1},t_i)}},
$$

$$
\rho=\frac{\sqrt{\varphi(t_0,t_{i-1},t_i)}}{\sqrt{g(t_{i-1},t_i)+\varphi(t_0,t_{i-1},t_i)}},
$$

$$p_0^r(t) = \frac{\ln\left[\frac{P(t,t_i,t_{i+1})}{(1+\delta r)^{-1}}\right] + \frac{1}{2}\left\{g(t_{i-1},t_i) + \left[\sqrt{\varphi(t,t_{i-1},t_{i+1})} - \sqrt{\varphi(t,t_{i-1},t_i)}\right]^2\right\}}{\sqrt{g(t_{i-1},t_i) + \left[\sqrt{\varphi(t,t_{i-1},t_{i+1})} - \sqrt{\varphi(t,t_{i-1},t_i)}\right]^2}},$$

$$p_1^r(t) = \frac{\ln\left[\frac{P(t,t_{i-1},t_i)}{(1+\delta r)^{-1}}\right] + \sqrt{\hat{h}(t_{i-1},t_i)\, g(t_{i-1},t_i)} - \frac{1}{2}\varphi(t,t_{i-1},t_i) + \sqrt{\varphi(t,t_{i-1},t_i)\,\varphi(t,t_{i-1},t_{i+1})}}{\sqrt{g(t_{i-1},t_i) + \varphi(t,t_{i-1},t_i)}},$$

$$\begin{aligned} p_2^r(t) = \quad & [g(t_{i-1},t_i) + \varphi(t,t_{i-1},t_i)]^{-\frac{1}{2}} \left\{\ln\left[\frac{P^{-1}(t,t_{i-1},t_i)}{(1+\delta r)^{-1}(1+\delta r_H)^2}\right] + \sqrt{\hat{h}(t_{i-1},t_i)\, g(t_{i-1},t_i)}\right. \\ & \left. + \left[\frac{1}{2} + 2\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\right]\varphi(t,t_{i-1},t_i) - \sqrt{\varphi(t,t_{i-1},t_i)\,\varphi(t,t_{i-1},t_{i+1})}\right\}, \end{aligned}$$

and where $r_H = r_u (> r_n(t_0, t_0+\delta))$ and $\theta = 1$ for a partial up-and-in caplet, or $r_H = r_l (< r_n(t_0, t_0+\delta))$ and $\theta = -1$ for a partial down-and-in floorlet.

Proof. See appendix D. ■

Remark 15 *As expected, it is easy to show that*

$$\lim_{t_{i-1}\to t_0} IN(\theta)_{t_0}^P [r_n(t_i,t_{i+1}); r_k; r_H; R=0; t_{i+1}] = IN(\theta)_{t_0}[r_n(t_i,t_{i+1}); r_k; r_H; R=0; t_{i+1}],$$

because $\varphi(t_0,t_0,T) = 0$ $(T \geq t_0)$ and $\lim_{t_{i-1}\to t_0}[-\theta z^*] = \infty$.

Proposition 8 *Under the Gaussian HJM specification (4), and subject to condition (22) and approximations (34) and (46),*

$$\begin{aligned} & OUT(\theta)_{t_0}^P [r_n(t_i,t_{i+1}); r_k; r_H; R=0; t_{i+1}] \qquad (52) \\ = \quad & \theta P(t_0,t_{i-1})\, P(t_0,t_i,t_{i+1}) \exp\left[\frac{1}{2}g(t_{i-1},t_i) - \sqrt{\hat{h}(t_{i-1},t_i)\, g(t_{i-1},t_i)} + \varphi(t_0,t_{i-1},t_i)\right. \\ & \left. - \sqrt{\varphi(t_0,t_{i-1},t_i)\,\varphi(t_0,t_{i-1},t_{i+1})}\right] M\left[z^* - \sqrt{\varphi(t_0,t_{i-1},t_{i+1})} + \sqrt{\varphi(t_0,t_{i-1},t_i)};\right. \\ & \left.\theta\sqrt{g(t_{i-1},t_i) + \varphi(t_0,t_{i-1},t_i)} - \theta p_1^{\theta\max(\theta r_H,\theta r_k)}(t_0); \rho\right] \\ & -\theta(1+\delta r_k)\, P(t_0,t_{i+1})\, M\left[z^* - \sqrt{\varphi(t_0,t_{i-1},t_{i+1})}; -\theta p_1^{\theta\max(\theta r_H,\theta r_k)}(t_0); \rho\right] \\ & -\theta P(t_0,t_{i+1})\, P(t_0,t_{i-1},t_i)\,(1+\delta r_H)^2 \exp\left\{2\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\ln\left[\frac{P^{-1}(t_0,t_{i-1},t_i)}{1+\delta r_H}\right] + \frac{g(t_{i-1},t_i)}{2}\right. \\ & - \sqrt{\hat{h}(t_{i-1},t_i)\, g(t_{i-1},t_i)} + \left[1 - 2\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\right]\sqrt{\varphi(t_0,t_{i-1},t_i)}\left[\sqrt{\varphi(t_0,t_{i-1},t_{i+1})}\right. \\ & \left.\left. - \sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\sqrt{\varphi(t_0,t_{i-1},t_i)}\right]\right\} M\left[z^* - \sqrt{\varphi(t_0,t_{i-1},t_{i+1})} - \left(1 - 2\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\right)\right. \\ & \left.\sqrt{\varphi(t_0,t_{i-1},t_i)}; \theta\sqrt{g(t_{i-1},t_i) + \varphi(t_0,t_{i-1},t_i)} - \theta p_2^{\theta\max(\theta r_H,\theta r_k)}(t_0); -\rho\right] \\ & +\theta(1+\delta r_k)\, P(t_0,t_{i+1}) \exp\left\{2\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\ln\left[\frac{P^{-1}(t_0,t_{i-1},t_i)}{1+\delta r_H}\right] + \sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\right. \end{aligned}$$

$$\sqrt{\varphi(t_0,t_{i-1},t_i)}\left[\left(1+2\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\right)\sqrt{\varphi(t_0,t_{i-1},t_i)}-2\sqrt{\varphi(t_0,t_{i-1},t_{i+1})}\right]\Bigg\}$$

$$M\left[z^*-\sqrt{\varphi(t_0,t_{i-1},t_{i+1})}+2\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\sqrt{\varphi(t_0,t_{i-1},t_i)};-\theta p_2^{\theta\max(\theta r_H,\theta r_k)}(t_0);-\rho\right],$$

where $r_H=r_l(<r_n(t_0,t_0+\delta))$ *and* $\theta=1$ *for a partial down-and-out caplet, or* $r_H=r_u(>r_n(t_0,t_0+\delta))$ *and* $\theta=-1$ *for a partial up-and-out floorlet.*

Proof. See appendix E. ■

4.4 Examples

Table 3 prices a two-year (unit face value) partial up-and-in cap with a cap rate of 6.045%, a barrier rate of 7% and quarterly compounding. Approximate partial up-and-in caplet prices were computed from proposition 7. The only exception was the first partial caplet (with a time-to-maturity of six months and a barrier monitoring period over the next three months), which is still given by proposition 3. Bivariate normal cumulative probabilities are computed from Drezner (1978), using the typo correction noticed in Hull (2000, page 272). Exact prices are computed through standard Monte Carlo simulation, with 520 time steps per year, 200,000 simulations, and assuming that the barrier is monitored 260 times per year (i.e., daily monitoring). The proposed approximation is still fast and produces pricing errors which decrease in option maturity. Although such pricing errors are rather large, notice that they are about 13 times smaller than the results based on regular caplet prices.

Table 4 considers a two-year (unit face value) partial down-and-out cap with a cap rate of 6.045%, a barrier rate of 6% and quarterly compounding. Exact prices are still obtained through standard Monte Carlo simulation under daily monitoring of the barrier, with 520 time steps per year, and through 200,000 simulations. Approximate partial down-and-out caplet prices were computed from proposition 8, with the exception of the first component caplet which is still valued through proposition 4. Again, such approximate solution is very fast and accurate, although the pricing errors increase with the option maturity.

5 Lookback caps and floors

5.1 Definitions

Lookback caps (floors) are sequences of caplets (floorlets), whose terminal payoff (with settlement in arrears) will depend on the maximum or on the minimum attained by the underlying "money-market" spot interest rate over a certain period of time. Two types of lookback options will be considered: *floating* and *fixed strike lookbacks*.

A floating strike lookback caplet (floorlet) is an European call (put) option on a spot nominal interest rate, whose strike is set equal to the minimum (maximum) value attained by such underlying interest rate over the life of the caplet (floorlet). Hence, these contracts are *no-regret options* in the sense that they will never finish out-of-the-money.

Definition 5 *The time-*t_{i+1} *price of a unit face value floating strike lookback caplet (floating strike lookback floorlet) on the nominal spot rate* $r_n(t_i,t_{i+1})$*, and settled in arrears at time* t_{i+1} *is equal to*

$$L(\theta)_{t_{i+1}}\left[r_n(t_i,t_{i+1});\theta\sup_{t_0\leq s\leq t_i}[\theta r_n(s,s+\delta)];t_{i+1}\right]:=\left\{\sup_{t_0\leq s\leq t_i}[\theta r_n(s,s+\delta)]-\theta r_n(t_i,t_{i+1})\right\}\delta, \quad (53)$$

where $\theta=-1$ *for a floating strike lookback caplet, or* $\theta=1$ *for a floating strike lookback floorlet.*

A fixed strike lookback caplet (floorlet) is an European call (put) option on the maximum (minimum) attained by a spot nominal interest rate over the life of the caplet (floorlet). For these exotic options, the strike is known since inception.

Definition 6 *The time-t_{i+1} price of a unit face value fixed strike lookback caplet (fixed strike lookback floorlet) on the δ-years nominal spot interest rate, with a cap (floor) rate of r_k, and settled in arrears at time t_{i+1} is equal to*

$$L(\theta)_{t_{i+1}}\left[\theta \sup_{t_0\leq s\leq t_i}\left[\theta r_n(s,s+\delta)\right];r_k;t_{i+1}\right]:=\left\{\sup_{t_0\leq s\leq t_i}\left[\theta r_n(s,s+\delta)\right]-\theta r_k\right\}^+\delta, \tag{54}$$

where $\theta=1$ for a fixed strike lookback caplet, or $\theta=-1$ for a fixed strike lookback floorlet.

Applying identity (6) to definitions (53) and (54), the time-t_0 price of the above mentioned lookback options can be written as

$$\begin{aligned}&L(\theta)_{t_0}\left[r_n(t_i,t_{i+1});\theta\sup_{t_0\leq s\leq t_i}\left[\theta r_n(s,s+\delta)\right];t_{i+1}\right] \qquad (55)\\ =\;& P(t_0,t_{i+1})E_{t_0}^{\mathcal{Q}^{t_{i+1}}}\left\{\theta\exp\left[\theta\sup_{t_0\leq s\leq t_i}\left(\theta\ln P^{-1}(s,s+\delta)\right)\right]-\theta\exp\left[\ln P^{-1}(t_i,t_{i+1})\right]\right\},\end{aligned}$$

and

$$\begin{aligned}&L(\theta)_{t_0}\left[\theta\sup_{t_0\leq s\leq t_i}\left[\theta r_n(s,s+\delta)\right];r_k;t_{i+1}\right] \qquad (56)\\ =\;& P(t_0,t_{i+1})E_{t_0}^{\mathcal{Q}^{t_{i+1}}}\left\{\left[\theta\exp\left(\theta\sup_{t_0\leq s\leq t_i}\left(\theta\ln P^{-1}(s,s+\delta)\right)\right)-\theta(1+\delta r_k)\right]^+\right\}.\end{aligned}$$

5.2 Densities for log-inverse *extrema* pure discount bond prices

In order to compute, explicitly, the expectations contained in equations (55) and (56), it is necessary to derive the probability density functions for the maximum and for the minimum of log-inverse discount factors. Such preliminary result will be easily extracted through integration over the restricted density function offered by theorem 1.

Corollary 2 *Under the Gaussian HJM specification (4), and subject to condition (22) and approximation (34),*

$$\begin{aligned}&\Pr_{\mathcal{Q}^{t_{i+1}}}\left\{\theta\sup_{t_0\leq s\leq t}\left[\theta\ln P^{-1}(s,s+\delta)\right]\in dy\middle|\mathcal{F}_{t_0}\right\} \qquad (57)\\ =\;&\Bigg\{\phi\left[y;\ln P^{-1}(t_0,t_0+\delta)-\sqrt{h(t_0,t)g(t_0,t)},\sqrt{g(t_0,t)}\right]\\ &+2\theta\sqrt{\frac{h(t_0,t)}{g(t_0,t)}}\exp\left[2\sqrt{\frac{h(t_0,t)}{g(t_0,t)}}\ln\frac{P^{-1}(t_0,t_0+\delta)}{y}\right]\Phi\left[\theta\frac{\ln\left(\frac{P^{-1}(t_0,t_0+\delta)}{y}\right)+\sqrt{h(t_0,t)g(t_0,t)}}{\sqrt{g(t_0,t)}}\right]\\ &+\exp\left[2\sqrt{\frac{h(t_0,t)}{g(t_0,t)}}\ln\frac{P^{-1}(t_0,t_0+\delta)}{y}\right]\phi\left[y;\ln P^{-1}(t_0,t_0+\delta)+\sqrt{h(t_0,t)g(t_0,t)},\sqrt{g(t_0,t)}\right]\Bigg\}dy,\end{aligned}$$

where $\theta=\{-1,1\}$.

Proof. Integrating density (23) over the interval $]-\infty,x]$, for $\theta=1$, or over $[x,+\infty[$, for $\theta=-1$,

$$\begin{aligned}&\Pr_{\mathcal{Q}^{t_{i+1}}}\left\{\theta\ln P^{-1}(t,t+\delta)\leq\theta x\wedge\sup_{t_0\leq s\leq t}\left[\theta\ln P^{-1}(s,s+\delta)\right]<\theta y\middle|\mathcal{F}_{t_0}\right\} \qquad (58)\\ \approx\;&\Phi\left[\theta\frac{x-\ln P^{-1}(t_0,t_0+\delta)+\sqrt{h(t_0,t)g(t_0,t)}}{\sqrt{g(t_0,t)}}\right]-\exp\left\{2\sqrt{\frac{h(t_0,t)}{g(t_0,t)}}\ln\left[\frac{P^{-1}(t_0,t_0+\delta)}{y}\right]\right\}\\ &\Phi\left[\theta\frac{x-2y+\ln P^{-1}(t_0,t_0+\delta)+\sqrt{h(t_0,t)g(t_0,t)}}{\sqrt{g(t_0,t)}}\right].\end{aligned}$$

Finally, replacing x by y in equation (58), and differentiating it with respect to y yields the approximate analytical solution (57). ■

5.3 Pricing formulae

Next proposition provides an explicit solution for equation (55) using both the "extrema" and the "unrestricted" densities (57) and (15), respectively.

Proposition 9 *Under the Gaussian HJM specification (4), and subject to condition (22) and approximation (34),*

$$L(\theta)_{t_0}\left[r_n(t_i,t_{i+1});\theta\sup_{t_0\leq s\leq t_i}[\theta r_n(s,s+\delta)];t_{i+1}\right] \tag{59}$$

$$= \theta P(t_0,t_{i+1})\,m(\theta)_{t_0}\,\Phi\left[-\theta l_1^\theta(t_0)\right]-\theta P(t_0,t_i)+\theta\left\{1+\left[1-2\sqrt{\frac{h(t_0,t_i)}{g(t_0,t_i)}}\right]^{-1}\right\}$$

$$\frac{P(t_0,t_{i+1})}{P(t_0,t_0+\delta)}\exp\left[\frac{1}{2}g(t_0,t_i)-\sqrt{h(t_0,t_i)\,g(t_0,t_i)}\right]\Phi\left[\theta l_1^\theta(t_0)+\theta\sqrt{g(t_0,t_i)}\right]$$

$$-\theta\frac{P(t_0,t_{i+1})\,m(\theta)_{t_0}}{1-2\sqrt{\frac{h(t_0,t_i)}{g(t_0,t_i)}}}\exp\left\{2\sqrt{\frac{h(t_0,t_i)}{g(t_0,t_i)}}\ln\left[\frac{P^{-1}(t_0,t_0+\delta)}{m(\theta)_{t_0}}\right]\right\}\Phi\left[\theta l_2^\theta(t_0)\right],$$

with

$$\ln\left[m(\theta)_{t_0}\right]:=\theta\sup_{s\leq t_0}\left[\theta\ln P^{-1}(s,s+\delta)\right],$$

$$l_1^\theta(t)=\frac{\ln\left[\frac{P^{-1}(t,t+\delta)}{m(\theta)_t}\right]-\sqrt{h(t,t_i)\,g(t,t_i)}}{\sqrt{g(t,t_i)}},$$

$$l_2^\theta(t)=\frac{\ln\left[\frac{P^{-1}(t,t+\delta)}{m(\theta)_t}\right]+\sqrt{h(t,t_i)\,g(t,t_i)}}{\sqrt{g(t,t_i)}},$$

and where $\theta=-1$ for a floating strike lookback caplet, or $\theta=1$ for a floating strike lookback floorlet.

Proof. See appendix F. ■

Remark 16 *The variable $m(\theta)_{t_0}$ represents, at time t_0, the running minimum (if $\theta=-1$) or maximum (if $\theta=1$) inverse pure discount price. Taking time-t_0 as the contract' inception date, then $m(\theta)_{t_0}=P^{-1}(t_0,t_0+\delta)$, $\quad\theta=-1,1$.*

Similarly, it is also possible to derive an approximate analytical pricing solution for the fixed strike lookback options specified in definition 6.

Proposition 10 *Under the Gaussian HJM specification (4), and subject to condition (22) and approximation (34),*

$$L(\theta)_{t_0}\left[\theta\sup_{t_0\leq s\leq t_i}[\theta r_n(s,s+\delta)];r_k;t_{i+1}\right] \tag{60}$$

$$= P(t_0,t_{i+1})\left[\theta r_n(t_0,t_0+\delta)-\theta r_k\right]^+\delta-\theta P(t_0,t_{i+1})\left\{1+\delta\theta\max\left[\theta r_k,\theta r_n(t_0,t_0+\delta)\right]\right\}\Phi\left[\theta l_3^\theta(t_0)\right]$$

$$+\theta\left\{1+\left[1-2\sqrt{\frac{h(t_0,t_i)}{g(t_0,t_i)}}\right]^{-1}\right\}\frac{P(t_0,t_{i+1})}{P(t_0,t_0+\delta)}\exp\left[\frac{1}{2}g(t_0,t_i)-\sqrt{h(t_0,t_i)\,g(t_0,t_i)}\right]$$

$$\Phi\left[\theta l_3^\theta(t_0)+\theta\sqrt{g(t_0,t_i)}\right]-\theta\frac{P(t_0,t_{i+1})\left\{1+\delta\theta\max\left[\theta r_k,\theta r_n(t_0,t_0+\delta)\right]\right\}}{1-2\sqrt{\frac{h(t_0,t_i)}{g(t_0,t_i)}}}$$

$$\exp\left\{2\sqrt{\frac{h(t_0,t_i)}{g(t_0,t_i)}}\ln\left[\frac{P^{-1}(t_0,t_0+\delta)}{1+\delta\theta\max\left(\theta r_k,\theta r_n(t_0,t_0+\delta)\right)}\right]\right\}\Phi\left[\theta l_4^\theta(t_0)\right],$$

with

$$l_3^\theta(t) = \frac{\ln P^{-1}(t, t+\delta) - \ln\left[1 + \delta\theta \max\left(\theta r_k, \theta r_n(t, t+\delta)\right)\right] - \sqrt{h(t, t_i)\, g(t, t_i)}}{\sqrt{g(t, t_i)}},$$

$$l_4^\theta(t) = \frac{\ln P^{-1}(t, t+\delta) - \ln\left[1 + \delta\theta \max\left(\theta r_k, \theta r_n(t, t+\delta)\right)\right] + \sqrt{h(t, t_i)\, g(t, t_i)}}{\sqrt{g(t, t_i)}},$$

and where $\theta = 1$ for a fixed strike lookback caplet, or $\theta = -1$ for a fixed strike lookback floorlet.

Proof. See appendix G. ■

5.4 Examples

Table 5 prices a two-year (unit face value) floating strike lookback cap with quarterly compounding, using the same three-factor HJM model as in the previous examples. Approximate lookback caplet prices were obtained from proposition 9. Exact option prices are still computed through standard Monte Carlo simulation, with 520 time steps per year, 200,000 simulations, and assuming that the minimum of the underlying interest rate is monitored 260 times per year (i.e., daily monitoring). The proposed approximation is still fast and rather accurate (the exotic cap' percentage pricing error is less than one percent).

Table 6 values, under the same term structure model, a two-year (unit face value) fixed strike lookback cap with a cap rate of 6.045%, and quarterly compounding. Exact option prices are still obtained from the same Monte Carlo setup (with daily monitoring of the underlying maximum). Approximate fixed strike lookback caplet prices follow from proposition 10, and are shown to be relatively close to their exact counterparts. Prices for regular caplets with the same strike are shown for comparison.

6 Conclusions and further research

The main purpose and contribution of this paper consisted in deriving approximate analytical pricing formulas, under a multi-factor Gaussian HJM framework, for (zero-rebate) barrier and lookback caps or floors. Through the numerical experiments presented in the previous sections, such closed-form solutions were shown to be efficient and rather accurate.

In terms of future research, it is still necessary to derive error bounds for the proposed approximate pricing solutions. Moreover, because most of the path-dependent options traded in the financial markets do not assume continuous monitoring, it is also important to adapt the proposed pricing solutions for discrete monitoring, through the approach suggested by Broadie et al. (1997). Such correction would most probably reduce the pricing errors reported in the numerical experiments presented in this paper. Finally, a possible extension of the pricing methodology developed in this paper would be to consider the existence of rebates for (knocked-out or non knocked-in) barrier caps or floors, through the derivation of first-passage time densities for the underlying nominal spot rate, based on the approximation provided by theorem 1.

A Appendix: Proof of proposition 3

A.1 Case 1: $\theta r_H < \theta r_k$

If $\theta r_H < \theta r_k$, then

$$\theta \ln P^{-1}(t_i, t_{i+1}) \geq \theta \ln(1 + \delta r_k) \Longrightarrow \theta \ln P^{-1}(t_i, t_{i+1}) \geq \theta \ln(1 + \delta r_H).$$

Therefore, the expectation contained in equation (11) can be computed using the second density on the right-hand-side of equation (37):

$$\begin{aligned} & IN(\theta)_{t_0}\left[r_n(t_i, t_{i+1}); r_k; \theta r_H < \theta r_k; R = 0; t_{i+1}\right] \\ = \; & P(t_0, t_{i+1}) \int_{-\infty}^{\infty} \left[\theta \exp(x) - \theta(1 + \delta r_k)\right]^+ \phi\left\{x; \ln\left[\frac{P(t_0, t_i)}{P(t_0, t_{i+1})}\right] + l(t_0, t_i), \sqrt{g(t_0, t_i)}\right\} dx. \end{aligned} \quad (61)$$

Comparing equations (20) and (61),

$$IN(\theta)_{t_0}\left[r_n(t_i,t_{i+1});r_k;\theta r_H<\theta r_k;R=0;t_{i+1}\right]=REGULAR(\theta)_{t_0}\left[r_n(t_i,t_{i+1});r_k;t_{i+1}\right]. \tag{62}$$

A.2 Case 2: $\theta r_H \geq \theta r_k$

In this case, the exercise region $\theta \ln P^{-1}(t_i,t_{i+1}) \geq \theta \ln(1+\delta r_k)$ can be located bellow or above the barrier level $\ln(1+\delta r_H)$. Hence, both density functions on the right-hand-side of equation (37) must be applied to equation (11):

$$\begin{aligned}
& IN(\theta)_{t_0}\left[r_n(t_i,t_{i+1});r_k;\theta r_H\geq\theta r_k;R=0;t_{i+1}\right] \qquad (63)\\
= \; & P(t_0,t_{i+1})\,\theta\int_{\ln(1+\delta r_k)}^{\ln(1+\delta r_H)}\left[\theta\exp(x)-\theta(1+\delta r_k)\right]\Big\{\phi\left[x;\ln P^{-1}(t_0,t_i,t_{i+1})+l(t_0,t_i),\sqrt{g(t_0,t_i)}\right]\\
& -\phi\left[x;\ln P^{-1}(t_0,t_0+\delta)-\sqrt{h(t_0,t_i)\,g(t_0,t_i)},\sqrt{g(t_0,t_i)}\right]+\exp\left[2\sqrt{\frac{h(t_0,t_i)}{g(t_0,t_i)}}\ln\frac{P^{-1}(t_0,t_0+\delta)}{1+\delta r_H}\right]\\
& \phi\left[x;\ln(1+\delta r_H)^2-\ln P^{-1}(t_0,t_0+\delta)-\sqrt{h(t_0,t_i)\,g(t_0,t_i)},\sqrt{g(t_0,t_i)}\right]\Big\}\,dx\\
& +P(t_0,t_{i+1})\,\theta\int_{\ln(1+\delta r_H)}^{\theta\infty}\left[\theta\exp(x)-\theta(1+\delta r_k)\right]\phi\left[x;\ln P^{-1}(t_0,t_i,t_{i+1})+l(t_0,t_i),\sqrt{g(t_0,t_i)}\right]dx.
\end{aligned}$$

Adding the first and last terms on the right-hand-side of equation (63), and using result (20),

$$\begin{aligned}
& IN(\theta)_{t_0}\left[r_n(t_i,t_{i+1});r_k;\theta r_H\geq\theta r_k;R=0;t_{i+1}\right] \qquad (64)\\
= \; & REGULAR(\theta)_{t_0}\left[r_n(t_i,t_{i+1});r_k;t_{i+1}\right]\\
& +P(t_0,t_{i+1})\,\theta\int_{\ln(1+\delta r_k)}^{\ln(1+\delta r_H)}\left[\theta\exp(x)-\theta(1+\delta r_k)\right]\\
& \Big\{-\phi\left[x;\ln P^{-1}(t_0,t_0+\delta)-\sqrt{h(t_0,t_i)\,g(t_0,t_i)},\sqrt{g(t_0,t_i)}\right]+\exp\left[2\sqrt{\frac{h(t_0,t_i)}{g(t_0,t_i)}}\ln\frac{P^{-1}(t_0,t_0+\delta)}{1+\delta r_H}\right]\\
& \phi\left[x;-\ln\frac{P^{-1}(t_0,t_0+\delta)}{(1+\delta r_H)^2}-\sqrt{h(t_0,t_i)\,g(t_0,t_i)},\sqrt{g(t_0,t_i)}\right]\Big\}\,dx.
\end{aligned}$$

Finally, equation (38) arises by noticing that $\theta^2=1$, using definition

$$\Phi\left(\frac{x-\mu}{\sigma}\right)=\int_{-\infty}^{x}\phi(z;\mu,\sigma)\,dz, \tag{65}$$

and applying the following well known lemma.[16]

Lemma 2 *If $z \backsim N^1(\mu,\sigma^2)$, then*

$$\int_{-\infty}^{x}\exp(z)\,\phi(z;\mu,\sigma)\,dz=\exp\left(\frac{1}{2}\sigma^2+\mu\right)\Phi\left(\frac{x-\mu}{\sigma}-\sigma\right). \tag{66}$$

■

B Appendix: Proof of proposition 4

B.1 Case 1: $\theta r_H < \theta r_k$

If $\theta r_H < \theta r_k$, then

$$\theta\ln P^{-1}(t_i,t_{i+1})>\theta\ln(1+\delta r_k)\Longrightarrow\theta\ln P^{-1}(t_i,t_{i+1})>\theta\ln(1+\delta r_H).$$

[16] See, for instance, Ingersoll (1987, page 14).

Therefore, the expectation contained in equation (12) can be computed using the probability density offered by theorem 1:

$$\begin{aligned}
& OUT(\theta)_{t_0}\left[r_n(t_i,t_{i+1});r_k;\theta r_H<\theta r_k;R=0;t_{i+1}\right] \qquad (67)\\
= \; & P(t_0,t_{i+1})\,\theta\int_{\ln(1+\delta r_k)}^{\theta\infty}\left[\theta\exp(x)-\theta(1+\delta r_k)\right]\\
& \left\{\phi\left[x;\ln P^{-1}(t_0,t_0+\delta)-\sqrt{h(t_0,t_i)\,g(t_0,t_i)},\sqrt{g(t_0,t_i)}\right]-\exp\left[2\sqrt{\frac{h(t_0,t_i)}{g(t_0,t_i)}}\ln\frac{P^{-1}(t_0,t_0+\delta)}{1+\delta r_H}\right]\right.\\
& \left.\phi\left[x;-\ln\frac{P^{-1}(t_0,t_0+\delta)}{(1+\delta r_H)^2}-\sqrt{h(t_0,t_i)\,g(t_0,t_i)},\sqrt{g(t_0,t_i)}\right]\right\}dx.
\end{aligned}$$

Using definition (65), the symmetry property $\Phi(-x)=1-\Phi(x)$, and lemma 2,

$$\begin{aligned}
& OUT(\theta)_{t_0}\left[r_n(t_i,t_{i+1});r_k;\theta r_H<\theta r_k;R=0;t_{i+1}\right] \qquad (68)\\
= \; & \theta P(t_0,t_{i+1})\,P^{-1}(t_0,t_0+\delta)\exp\left[\frac{1}{2}g(t_0,t_i)-\sqrt{h(t_0,t_i)\,g(t_0,t_i)}\right]\Phi\left[\theta\sqrt{g(t_0,t_i)}-\theta d_1^{r_k}(t_0)\right]\\
& -\theta(1+\delta r_k)\,P(t_0,t_{i+1})\,\Phi\left[-\theta d_1^{r_k}(t_0)\right]-\theta\exp\left\{2\sqrt{\frac{h(t_0,t_i)}{g(t_0,t_i)}}\ln\left[\frac{P^{-1}(t_0,t_0+\delta)}{1+\delta r_H}\right]\right\}\\
& \left\{P(t_0,t_{i+1})\,P(t_0,t_0+\delta)\,(1+\delta r_H)^2\exp\left[\frac{1}{2}g(t_0,t_i)-\sqrt{h(t_0,t_i)\,g(t_0,t_i)}\right]\Phi\left[\theta\sqrt{g(t_0,t_i)}-\theta d_2^{r_k}(t_0)\right]\right.\\
& \left.-(1+\delta r_k)\,P(t_0,t_{i+1})\,\Phi\left[-\theta d_2^{r_k}(t_0)\right]\right\},
\end{aligned}$$

which is in accordance with equation (39), for $\theta r_H<\theta r_k$.

B.2 Case 2: $\theta r_H \geq \theta r_k$

Since the *down-and-out caplet* (*up-and-out floorlet*) finishes worthless if $\ln P^{-1}(t_i,t_i+\delta)\in[\ln(1+\delta r_k),\ln(1+\delta r_l)]$ ($\ln P^{-1}(t_i,t_i+\delta)\in[\ln(1+\delta r_u),\ln(1+\delta r_k)]$), applying again theorem 1 to equation (12),

$$\begin{aligned}
& OUT(\theta)_{t_0}\left[r_n(t_i,t_{i+1});r_k;\theta r_H\geq\theta r_k;R=0;t_{i+1}\right] \qquad (69)\\
= \; & P(t_0,t_{i+1})\,\theta\int_{\ln(1+\delta r_H)}^{\theta\infty}\left[\theta\exp(x)-\theta(1+\delta r_k)\right]\\
& \left\{\phi\left[x;\ln P^{-1}(t_0,t_0+\delta)-\sqrt{h(t_0,t_i)\,g(t_0,t_i)},\sqrt{g(t_0,t_i)}\right]-\exp\left[2\sqrt{\frac{h(t_0,t_i)}{g(t_0,t_i)}}\ln\frac{P^{-1}(t_0,t_0+\delta)}{1+\delta r_H}\right]\right.\\
& \left.\phi\left[x;-\ln\frac{P^{-1}(t_0,t_0+\delta)}{(1+\delta r_H)^2}-\sqrt{h(t_0,t_i)\,g(t_0,t_i)},\sqrt{g(t_0,t_i)}\right]\right\}dx.
\end{aligned}$$

Comparing the right-hand side of equations (67) and (69), it follows that equation (69) is equivalent to solution (68) with $d_1^{r_k}(t_0)$ and $d_2^{r_k}(t_0)$ replaced by $d_1^{r_H}(t_0)$ and $d_2^{r_H}(t_0)$, respectively, which agrees with equation (39), for $\theta r_H\geq\theta r_k$. ■

C Appendix: Gauss-Markov time-homogeneous specification

Next proposition recasts definitions (17), (24) and (48) under a Markovian time-independent HJM model.

Proposition 11 *Under the volatility specification of proposition 5,*

$$g(t,t_i)=\underline{\sigma}(\delta)'\cdot\Delta(t_i-t)\cdot\underline{\sigma}(\delta), \qquad (70)$$

$$\frac{\partial g(t,v)}{\partial v} = \left\|\underline{\sigma}(\delta)\right\|^2 + 2\underline{G}' \cdot \left(I_n - e^{a\delta}\right) \cdot \Delta(v-t) \cdot \underline{\sigma}(\delta), \tag{71}$$

$$\begin{aligned}\frac{\partial l(t,v)}{\partial v} = \quad & \frac{1}{2}\left\|\underline{\sigma}(\delta)\right\|^2 - \underline{\sigma}(\delta)' \cdot \underline{\sigma}(t_{i+1}-v) + \underline{G}' \cdot a^{-1} \cdot \Big[e^{a\delta} \cdot \Delta(v-t) \cdot e^{a'\delta} \\ & -\Delta(v-t) + e^{a(t_{i+1}-v)} \cdot \Delta(v-t) \cdot \left(I_n - e^{a'\delta}\right)\Big] \cdot \underline{G},\end{aligned} \tag{72}$$

and

$$\varphi(t_0, t_{i-1}, T) = \underline{\sigma}(T-t_{i-1})' \cdot \Delta(t_{i-1}-t_0) \cdot \underline{\sigma}(T-t_{i-1}), \tag{73}$$

with

$$\Delta(T-t) = (a+a')^{-1} \cdot \left[e^{(a+a')(T-t)} - I_n\right]. \tag{74}$$

Proof. Available upon request. ■

Remark 17 *Function $h(t,t_i)$ can now be computed using solutions (71) and (72), through Romberg's integration. Concerning the matrix exponentials involved in the previous formulae, these are computed using Padé approximations with scaling and squaring. For details, see Van Loan (1978).*

D Appendix: Proof of proposition 7

Replacing t_0 by t_{i-1} in equations (19) and (38), and applying proposition 6 as well as approximation (46), then equation (43) can be rewritten as:

$$\begin{aligned}& IN(\theta)_{t_0}^{P}\left[r_n(t_i,t_{i+1}); r_k; r_H; R=0; t_{i+1}\right] \\ \approx\ & \theta P(t_0,t_i) \int_{\Re} (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}z^2\right) \exp\left[-\frac{1}{2}\varphi(t_0,t_{i-1},t_i) + \sqrt{\varphi(t_0,t_{i-1},t_i)}z\right] \\ & \Phi\left\{-\theta\frac{\ln\left[\frac{P(t_0,t_i,t_{i+1})}{(1+\delta r_k)^{-1}}\right] - \frac{1}{2}\left[g(t_{i-1},t_i) + \varphi(t,t_{i-1},t_{i+1}) - \varphi(t,t_{i-1},t_i)\right]}{\sqrt{g(t_{i-1},t_i)}}\right. \\ & \left. -\theta\frac{\sqrt{\varphi(t_0,t_{i-1},t_{i+1})} - \sqrt{\varphi(t_0,t_{i-1},t_i)}}{\sqrt{g(t_{i-1},t_i)}}z\right\} dz \\ & -\theta(1+\delta r_k) P(t_0,t_{i+1}) \int_{\Re} (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}z^2\right) \exp\left[-\frac{1}{2}\varphi(t_0,t_{i-1},t_{i+1}) + \sqrt{\varphi(t_0,t_{i-1},t_{i+1})}z\right] \\ & \Phi\left\{-\theta\frac{\ln\left[\frac{P(t_0,t_i,t_{i+1})}{(1+\delta r_k)^{-1}}\right] + \frac{1}{2}\left[g(t_{i-1},t_i) - \varphi(t,t_{i-1},t_{i+1}) + \varphi(t,t_{i-1},t_i)\right]}{\sqrt{g(t_{i-1},t_i)}}\right. \\ & \left. -\theta\frac{\sqrt{\varphi(t_0,t_{i-1},t_{i+1})} - \sqrt{\varphi(t_0,t_{i-1},t_i)}}{\sqrt{g(t_{i-1},t_i)}}z\right\} dz \\ & +1_{\{\theta r_H > \theta r_k\}}(1+\delta r_k) P(t_0,t_{i+1}) \int_{\Re} \frac{1_{\{\theta r_n(t_{i-1},t_i) < \theta r_H\}}}{\sqrt{2\pi}} \exp\left[-\frac{\varphi(t_0,t_{i-1},t_{i+1})}{2} + \sqrt{\varphi(t_0,t_{i-1},t_{i+1})}z\right. \\ & \left. -\frac{1}{2}z^2\right] \left\{\Phi\left[\frac{\ln\left(\frac{P(t_0,t_{i-1},t_i)}{(1+\delta r_H)^{-1}}\right) + \sqrt{\hat{h}(t_{i-1},t_i) g(t_{i-1},t_i)} - \frac{1}{2}\varphi(t_0,t_{i-1},t_i)}{\sqrt{g(t_{i-1},t_i)}} + \sqrt{\frac{\varphi(t_0,t_{i-1},t_i)}{g(t_{i-1},t_i)}}z\right]\right. \\ & \left. -\Phi\left[\frac{\ln\left(\frac{P(t_0,t_{i-1},t_i)}{(1+\delta r_k)^{-1}}\right) + \sqrt{\hat{h}(t_{i-1},t_i) g(t_{i-1},t_i)} - \frac{1}{2}\varphi(t_0,t_{i-1},t_i)}{\sqrt{g(t_{i-1},t_i)}} + \sqrt{\frac{\varphi(t_0,t_{i-1},t_i)}{g(t_{i-1},t_i)}}z\right]\right\} dz \\ & -1_{\{\theta r_H > \theta r_k\}} P(t_0,t_{i-1}) P(t_0,t_i,t_{i+1}) \exp\left[\frac{g(t_{i-1},t_i)}{2} - \sqrt{\hat{h}(t_{i-1},t_i) g(t_{i-1},t_i)}\right] \int_{\Re} \frac{1_{\{\theta r_n(t_{i-1},t_i) < \theta r_H\}}}{\sqrt{2\pi}}\end{aligned} \tag{75}$$

$$\exp\left\{-\frac{1}{2}z^2-\frac{1}{2}\left[\varphi\left(t_0,t_{i-1},t_{i+1}\right)-\varphi\left(t_0,t_{i-1},t_i\right)\right]+\left[\sqrt{\varphi\left(t_0,t_{i-1},t_{i+1}\right)}-\sqrt{\varphi\left(t_0,t_{i-1},t_i\right)}\right]z\right\}$$
$$\left\{\Phi\left[\frac{\ln\left(\frac{P(t_0,t_{i-1},t_i)}{(1+\delta r_H)^{-1}}\right)+\sqrt{\hat{h}\left(t_{i-1},t_i\right)g\left(t_{i-1},t_i\right)}-\frac{1}{2}\varphi\left(t_0,t_{i-1},t_i\right)-g\left(t_{i-1},t_i\right)}{\sqrt{g\left(t_{i-1},t_i\right)}}+\sqrt{\frac{\varphi\left(t_0,t_{i-1},t_i\right)}{g\left(t_{i-1},t_i\right)}}z\right]\right.$$
$$\left.-\Phi\left[\frac{\ln\left(\frac{P(t_0,t_{i-1},t_i)}{(1+\delta r_k)^{-1}}\right)+\sqrt{\hat{h}\left(t_{i-1},t_i\right)g\left(t_{i-1},t_i\right)}-\frac{\varphi(t_0,t_{i-1},t_i)}{2}-g\left(t_{i-1},t_i\right)}{\sqrt{g\left(t_{i-1},t_i\right)}}+\sqrt{\frac{\varphi\left(t_0,t_{i-1},t_i\right)}{g\left(t_{i-1},t_i\right)}}z\right]\right\}dz$$
$$+1_{\{\theta r_H>\theta r_k\}}P\left(t_0,t_{i+1}\right)P\left(t_0,t_{i-1},t_i\right)\left(1+\delta r_H\right)^2\exp\left\{2\sqrt{\frac{\hat{h}\left(t_{i-1},t_i\right)}{g\left(t_{i-1},t_i\right)}}\ln\left[\frac{P^{-1}\left(t_0,t_{i-1},t_i\right)}{1+\delta r_H}\right]\right\}$$
$$\exp\left[\frac{1}{2}g\left(t_{i-1},t_i\right)-\sqrt{\hat{h}\left(t_{i-1},t_i\right)g\left(t_{i-1},t_i\right)}\right]\int_{\Re}\frac{1_{\{\theta r_n(t_{i-1},t_i)<\theta r_H\}}}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}z^2\right)$$
$$\exp\left\{-\frac{1}{2}\left[\varphi\left(t_0,t_{i-1},t_{i+1}\right)+\varphi\left(t_0,t_{i-1},t_i\right)\right]+\left[\sqrt{\varphi\left(t_0,t_{i-1},t_{i+1}\right)}+\sqrt{\varphi\left(t_0,t_{i-1},t_i\right)}\right]z\right\}$$
$$\exp\left\{2\sqrt{\frac{\hat{h}\left(t_{i-1},t_i\right)}{g\left(t_{i-1},t_i\right)}}\left[\frac{1}{2}\varphi\left(t_0,t_{i-1},t_i\right)-\sqrt{\varphi\left(t_0,t_{i-1},t_i\right)}z\right]\right\}$$
$$\left\{\Phi\left[\frac{\ln\left(\frac{P^{-1}(t_0,t_{i-1},t_i)}{1+\delta r_H}\right)+\sqrt{\hat{h}\left(t_{i-1},t_i\right)g\left(t_{i-1},t_i\right)}+\frac{\varphi(t_0,t_{i-1},t_i)}{2}-g\left(t_{i-1},t_i\right)}{\sqrt{g\left(t_{i-1},t_i\right)}}-\sqrt{\frac{\varphi\left(t_0,t_{i-1},t_i\right)}{g\left(t_{i-1},t_i\right)}}z\right]\right.$$
$$\left.-\Phi\left[\frac{\ln\left(\frac{P^{-1}(t_0,t_{i-1},t_i)}{\frac{(1+\delta r_H)^2}{(1+\delta r_k)}}\right)+\sqrt{\hat{h}\left(t_{i-1},t_i\right)g\left(t_{i-1},t_i\right)}+\frac{\varphi(t_0,t_{i-1},t_i)}{2}-g\left(t_{i-1},t_i\right)}{\sqrt{g\left(t_{i-1},t_i\right)}}-\sqrt{\frac{\varphi\left(t_0,t_{i-1},t_i\right)}{g\left(t_{i-1},t_i\right)}}z\right]\right\}dz$$
$$-1_{\{\theta r_H>\theta r_k\}}\frac{P\left(t_0,t_{i+1}\right)}{\left(1+\delta r_k\right)^{-1}}\exp\left\{2\sqrt{\frac{\hat{h}\left(t_{i-1},t_i\right)}{g\left(t_{i-1},t_i\right)}}\ln\left[\frac{P^{-1}\left(t_0,t_{i-1},t_i\right)}{1+\delta r_H}\right]\right\}\int_{\Re}\frac{1_{\{\theta r_n(t_{i-1},t_i)<\theta r_H\}}}{\sqrt{2\pi}}\exp\left\{-\frac{z^2}{2}\right.$$
$$\left.+2\sqrt{\frac{\hat{h}\left(t_{i-1},t_i\right)}{g\left(t_{i-1},t_i\right)}}\left[\frac{\varphi\left(t_0,t_{i-1},t_i\right)}{2}-\sqrt{\varphi\left(t_0,t_{i-1},t_i\right)}z\right]-\left[\frac{\varphi\left(t_0,t_{i-1},t_{i+1}\right)}{2}-\sqrt{\varphi\left(t_0,t_{i-1},t_{i+1}\right)}z\right]\right\}$$
$$\left\{\Phi\left[\frac{\ln\left(\frac{P^{-1}(t_0,t_{i-1},t_i)}{1+\delta r_H}\right)+\sqrt{\hat{h}\left(t_{i-1},t_i\right)g\left(t_{i-1},t_i\right)}+\frac{1}{2}\varphi\left(t_0,t_{i-1},t_i\right)}{\sqrt{g\left(t_{i-1},t_i\right)}}-\sqrt{\frac{\varphi\left(t_0,t_{i-1},t_i\right)}{g\left(t_{i-1},t_i\right)}}z\right]\right.$$
$$\left.-\Phi\left[\frac{\ln\left(\frac{P^{-1}(t_0,t_{i-1},t_i)}{(1+\delta r_k)^{-1}(1+\delta r_H)^2}\right)+\sqrt{\hat{h}\left(t_{i-1},t_i\right)g\left(t_{i-1},t_i\right)}+\frac{1}{2}\varphi\left(t_0,t_{i-1},t_i\right)}{\sqrt{g\left(t_{i-1},t_i\right)}}-\sqrt{\frac{\varphi\left(t_0,t_{i-1},t_i\right)}{g\left(t_{i-1},t_i\right)}}z\right]\right\}dz.$$

Because

$$\theta r_n\left(t_{i-1},t_i\right)<\theta r_H\Longleftrightarrow\theta z>\theta z^*,$$

completing the squares in z, inside the exponential functions, and performing obvious changes of variables,

$$IN\left(\theta\right)_{t_0}^{P}\left[r_n\left(t_i,t_{i+1}\right);r_k;r_H;R=0;t_{i+1}\right] \tag{76}$$
$$\approx\quad\theta P\left(t_0,t_i\right)\int_{\Re}\left(2\pi\right)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}w^2\right)$$

$$\Phi\left\{\theta\left(\sqrt{g(t_{i-1},t_i)+\left[\sqrt{\varphi(t_0,t_{i-1},t_{i+1})}-\sqrt{\varphi(t_0,t_{i-1},t_i)}\right]^2}-p_0^{r_k}(t_0)-\rho_0 w\right)\frac{1}{\sqrt{1-\rho_0^2}}\right\}dw$$

$$-\theta(1+\delta r_k)P(t_0,t_{i+1})\int_{\Re}(2\pi)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}w^2\right)\Phi\left\{\theta\left[-p_0^{r_k}(t_0)-\rho_0 w\right]\frac{1}{\sqrt{1-\rho_0^2}}\right\}dw$$

$$+1_{\{\theta r_H>\theta r_k\}}(1+\delta r_k)P(t_0,t_{i+1})\,\theta\int_{-\theta\infty}^{-\theta z^*+\theta\sqrt{\varphi(t_0,t_{i-1},t_{i+1})}}(2\pi)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}w^2\right)$$

$$\left\{\Phi\left[\left(p_1^{r_H}(t_0)-\rho w\right)\frac{1}{\sqrt{1-\rho^2}}\right]-\Phi\left[\left(p_1^{r_k}(t_0)-\rho w\right)\frac{1}{\sqrt{1-\rho^2}}\right]\right\}dw$$

$$-\exp\left[\frac{1}{2}g(t_{i-1},t_i)-\sqrt{\hat{h}(t_{i-1},t_i)\,g(t_{i-1},t_i)}+\varphi(t_0,t_{i-1},t_i)-\sqrt{\varphi(t_0,t_{i-1},t_i)\,\varphi(t_0,t_{i-1},t_{i+1})}\right]$$

$$1_{\{\theta r_H>\theta r_k\}}P(t_0,t_{i-1})\,P(t_0,t_i,t_{i+1})\,\theta\int_{-\theta\infty}^{-\theta z^*+\theta\sqrt{\varphi(t_0,t_{i-1},t_{i+1})}-\theta\sqrt{\varphi(t_0,t_{i-1},t_i)}}(2\pi)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}w^2\right)$$

$$\left\{\Phi\left[\left(p_1^{r_H}(t_0)-\sqrt{g(t_{i-1},t_i)+\varphi(t_0,t_{i-1},t_i)}-\rho w\right)\frac{1}{\sqrt{1-\rho^2}}\right]\right.$$

$$\left.-\Phi\left[\left(p_1^{r_k}(t_0)-\sqrt{g(t_{i-1},t_i)+\varphi(t_0,t_{i-1},t_i)}-\rho w\right)\frac{1}{\sqrt{1-\rho^2}}\right]\right\}dw$$

$$+1_{\{\theta r_H>\theta r_k\}}P(t_0,t_{i+1})\,P(t_0,t_{i-1},t_i)\,(1+\delta r_H)^2\exp\left\{2\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\ln\left[\frac{P^{-1}(t_0,t_{i-1},t_i)}{1+\delta r_H}\right]\right\}$$

$$\exp\left\{\frac{1}{2}g(t_{i-1},t_i)-\sqrt{\hat{h}(t_{i-1},t_i)\,g(t_{i-1},t_i)}+\left[1-2\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\right]\sqrt{\varphi(t_0,t_{i-1},t_i)}\left[\sqrt{\varphi(t_0,t_{i-1},t_{i+1})}\right.\right.$$

$$\left.\left.-\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\sqrt{\varphi(t_0,t_{i-1},t_i)}\right]\right\}\theta\int_{-\theta\infty}^{-\theta z^*+\theta\sqrt{\varphi(t_0,t_{i-1},t_{i+1})}+\theta\left[1-2\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\right]\sqrt{\varphi(t_0,t_{i-1},t_i)}}(2\pi)^{-\frac{1}{2}}$$

$$\exp\left(-\frac{1}{2}w^2\right)\left\{\Phi\left[\left(p_2^{r_H}(t_0)-\sqrt{g(t_{i-1},t_i)+\varphi(t_0,t_{i-1},t_i)}+\rho w\right)\frac{1}{\sqrt{1-\rho^2}}\right]\right.$$

$$\left.-\Phi\left[\left(p_2^{r_k}(t_0)-\sqrt{g(t_{i-1},t_i)+\varphi(t_0,t_{i-1},t_i)}+\rho w\right)\frac{1}{\sqrt{1-\rho^2}}\right]\right\}dw$$

$$-1_{\{\theta r_H>\theta r_k\}}(1+\delta r_k)P(t_0,t_{i+1})\exp\left\{2\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\ln\left[\frac{P^{-1}(t_0,t_{i-1},t_i)}{1+\delta r_H}\right]\right\}$$

$$\exp\left\{\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\sqrt{\varphi(t_0,t_{i-1},t_i)}\left[\left(1+2\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\right)\sqrt{\varphi(t_0,t_{i-1},t_i)}-2\sqrt{\varphi(t_0,t_{i-1},t_{i+1})}\right]\right\}$$

$$\theta\int_{-\theta\infty}^{-\theta z^*+\theta\sqrt{\varphi(t_0,t_{i-1},t_{i+1})}-2\theta\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\sqrt{\varphi(t_0,t_{i-1},t_i)}}(2\pi)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}w^2\right)\left\{\Phi\left[\left(p_2^{r_H}(t_0)+\rho w\right)\frac{1}{\sqrt{1-\rho^2}}\right]\right.$$

$$\left.-\Phi\left[\left(p_2^{r_k}(t_0)+\rho w\right)\frac{1}{\sqrt{1-\rho^2}}\right]\right\}dw,$$

where

$$\rho_0=\frac{\sqrt{\varphi(t_0,t_{i-1},t_{i+1})}-\sqrt{\varphi(t_0,t_{i-1},t_i)}}{\sqrt{g(t_{i-1},t_i)+\left[\sqrt{\varphi(t_0,t_{i-1},t_{i+1})}-\sqrt{\varphi(t_0,t_{i-1},t_i)}\right]^2}}.$$

Finally, in order to convert equation (76) into the pricing solution (51), it is only necessary to apply the following lemma.

Lemma 3 *For* $a, b, \rho \in \Re$,

$$\int_{-\infty}^{a} (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}w^2\right) \Phi\left(\frac{b-\rho w}{\sqrt{1-\rho^2}}\right) dw = M(a;b;\rho), \tag{77}$$

$$\int_{a}^{\infty} (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}y^2\right) \Phi\left(\frac{b-\rho y}{\sqrt{1-\rho^2}}\right) dy = M(-a;b;-\rho), \tag{78}$$

$$\Phi(b) - M(a;b;\rho) = M(-a;b;-\rho), \tag{79}$$

and

$$\int_{-\infty}^{\infty} (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}w^2\right) \Phi\left(\frac{b-\rho w}{\sqrt{1-\rho^2}}\right) dw = \Phi(b). \tag{80}$$

Proof. Equation (77) is proved, for instance, in Geske (1979, page 80). Equation (78) follows from (77), when the change of variables $y = -w$ is applied. Relation (79) can be found, for instance, in Drezner (1978, equation 8). Identity (80) follows by replacing (77) and (78) into (79). ■

E Appendix: Proof of proposition 8

Similarly to the procedure followed in appendix D, replacing t_0 by t_{i-1} in equation (39), and applying proposition 6 as well as approximation (46), then equation (44) can be rewritten as:

$$\begin{aligned}
& OUT(\theta)_{t_0}^{P}\left[r_n(t_i,t_{i+1});r_k;r_H;R=0;t_{i+1}\right] \qquad (81)\\
\approx\ & \theta P(t_0,t_{i-1}) P(t_0,t_i,t_{i+1}) \exp\left[\frac{1}{2}g(t_{i-1},t_i) - \sqrt{\hat{h}(t_{i-1},t_i)\, g(t_{i-1},t_i)}\right] \int_{\Re} \frac{1_{\{\theta r_n(t_{i-1},t_i)>\theta r_H\}}}{\sqrt{2\pi}} \\
& \exp\left\{-\frac{1}{2}z^2 - \frac{1}{2}\left[\varphi(t_0,t_{i-1},t_{i+1}) - \varphi(t_0,t_{i-1},t_i)\right] + \left[\sqrt{\varphi(t_0,t_{i-1},t_{i+1})} - \sqrt{\varphi(t_0,t_{i-1},t_i)}\right] z\right\} \\
& \Phi\left\{-\theta\frac{\ln\left[\frac{P(t_0,t_{i-1},t_i)}{(1+\delta\theta\max(\theta r_H,\theta r_k))^{-1}}\right] + \sqrt{\hat{h}(t_{i-1},t_i)\, g(t_{i-1},t_i)} - \frac{1}{2}\varphi(t_0,t_{i-1},t_i) - g(t_{i-1},t_i)}{\sqrt{g(t_{i-1},t_i)}}\right. \\
& \left. -\theta\sqrt{\frac{\varphi(t_0,t_{i-1},t_i)}{g(t_{i-1},t_i)}}z\right\} dz \\
& -\theta(1+\delta r_k) P(t_0,t_{i+1}) \int_{\Re} \frac{1_{\{\theta r_n(t_{i-1},t_i)>\theta r_H\}}}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}z^2 - \frac{1}{2}\varphi(t_0,t_{i-1},t_{i+1}) + \sqrt{\varphi(t_0,t_{i-1},t_{i+1})}z\right] \\
& \Phi\left\{-\theta\frac{\ln\left[\frac{P(t_0,t_{i-1},t_i)}{(1+\delta\theta\max(\theta r_H,\theta r_k))^{-1}}\right] + \sqrt{\hat{h}(t_{i-1},t_i)\, g(t_{i-1},t_i)} - \frac{1}{2}\varphi(t_0,t_{i-1},t_i)}{\sqrt{g(t_{i-1},t_i)}} - \theta\sqrt{\frac{\varphi(t_0,t_{i-1},t_i)}{g(t_{i-1},t_i)}}z\right\} dz \\
& -\theta P(t_0,t_{i+1}) P(t_0,t_{i-1},t_i)(1+\delta r_H)^2 \exp\left\{2\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}} \ln\left[\frac{P^{-1}(t_0,t_{i-1},t_i)}{1+\delta r_H}\right]\right\} \\
& \exp\left[\frac{1}{2}g(t_{i-1},t_i) - \sqrt{\hat{h}(t_{i-1},t_i)\, g(t_{i-1},t_i)}\right] \int_{\Re} \frac{1_{\{\theta r_n(t_{i-1},t_i)>\theta r_H\}}}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right) \\
& \exp\left\{-\frac{1}{2}\left[\varphi(t_0,t_{i-1},t_{i+1}) + \varphi(t_0,t_{i-1},t_i)\right] + \left[\sqrt{\varphi(t_0,t_{i-1},t_{i+1})} + \sqrt{\varphi(t_0,t_{i-1},t_i)}\right] z\right\}
\end{aligned}$$

$$\exp\left\{2\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\left[\frac{1}{2}\varphi(t_0,t_{i-1},t_i)-\sqrt{\varphi(t_0,t_{i-1},t_i)}z\right]\right\}$$

$$\Phi\left\{-\theta\frac{\ln\left[\frac{P^{-1}(t_0,t_{i-1},t_i)}{(1+\delta\theta\max(\theta r_H,\theta r_k))^{-1}(1+\delta r_H)^2}\right]+\sqrt{\hat{h}(t_{i-1},t_i)\,g(t_{i-1},t_i)}+\frac{1}{2}\varphi(t_0,t_{i-1},t_i)-g(t_{i-1},t_i)}{\sqrt{g(t_{i-1},t_i)}}\right.$$

$$\left.+\theta\sqrt{\frac{\varphi(t_0,t_{i-1},t_i)}{g(t_{i-1},t_i)}}z\right\}dz$$

$$+\theta\frac{P(t_0,t_{i+1})}{(1+\delta r_k)^{-1}}\exp\left\{2\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\ln\left[\frac{P^{-1}(t_0,t_{i-1},t_i)}{1+\delta r_H}\right]\right\}\int_{\Re}\frac{1_{\{\theta r_n(t_{i-1},t_i)>\theta r_H\}}}{\sqrt{2\pi}}\exp\left\{-\frac{z^2}{2}\right.$$

$$\left.+2\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\left[\frac{\varphi(t_0,t_{i-1},t_i)}{2}-\sqrt{\varphi(t_0,t_{i-1},t_i)}z\right]-\left[\frac{\varphi(t_0,t_{i-1},t_{i+1})}{2}-\sqrt{\varphi(t_0,t_{i-1},t_{i+1})}z\right]\right\}$$

$$\Phi\left\{-\theta\frac{\ln\left[\frac{P^{-1}(t_0,t_{i-1},t_i)}{(1+\delta\theta\max(\theta r_H,\theta r_k))^{-1}(1+\delta r_H)^2}\right]+\sqrt{\hat{h}(t_{i-1},t_i)\,g(t_{i-1},t_i)}+\frac{1}{2}\varphi(t_0,t_{i-1},t_i)}{\sqrt{g(t_{i-1},t_i)}}\right.$$

$$\left.+\theta\sqrt{\frac{\varphi(t_0,t_{i-1},t_i)}{g(t_{i-1},t_i)}}z\right\}dz.$$

Because

$$\theta r_n(t_{i-1},t_i)>\theta r_H\Longleftrightarrow\theta z<\theta z^*,$$

completing the squares in z, inside the exponential functions, and performing obvious changes of variables,

$$\begin{aligned}
&OUT(\theta)_{t_0}^{P}[r_n(t_i,t_{i+1});r_k;r_H;R=0;t_{i+1}] \qquad (82)\\
\approx\;&\theta\exp\left[\frac{1}{2}g(t_{i-1},t_i)-\sqrt{\hat{h}(t_{i-1},t_i)\,g(t_{i-1},t_i)}+\varphi(t_0,t_{i-1},t_i)-\sqrt{\varphi(t_0,t_{i-1},t_i)\,\varphi(t_0,t_{i-1},t_{i+1})}\right]\\
&P(t_0,t_{i-1})\,P(t_0,t_i,t_{i+1})\,\theta\int_{-\theta\infty}^{\theta z^*-\theta\sqrt{\varphi(t_0,t_{i-1},t_{i+1})}+\theta\sqrt{\varphi(t_0,t_{i-1},t_i)}}(2\pi)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}w^2\right)\\
&\Phi\left\{\left[-\theta p_1^{\theta\max(\theta r_H,\theta r_k)}(t_0)+\theta\sqrt{g(t_{i-1},t_i)+\varphi(t_0,t_{i-1},t_i)}-\theta\rho w\right]\frac{1}{\sqrt{1-\rho^2}}\right\}dw\\
&-\theta(1+\delta r_k)\,P(t_0,t_{i+1})\,\theta\int_{-\theta\infty}^{\theta z^*-\theta\sqrt{\varphi(t_0,t_{i-1},t_{i+1})}}(2\pi)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}w^2\right)\\
&\Phi\left\{\left[-\theta p_1^{\theta\max(\theta r_H,\theta r_k)}(t_0)-\theta\rho w\right]\frac{1}{\sqrt{1-\rho^2}}\right\}dw\\
&-\theta P(t_0,t_{i+1})\,P(t_0,t_{i-1},t_i)\,(1+\delta r_H)^2\exp\left\{2\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\ln\left[\frac{P^{-1}(t_0,t_{i-1},t_i)}{1+\delta r_H}\right]\right\}\\
&\exp\left\{\frac{1}{2}g(t_{i-1},t_i)-\sqrt{\hat{h}(t_{i-1},t_i)\,g(t_{i-1},t_i)}+\left[1-2\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\right]\sqrt{\varphi(t_0,t_{i-1},t_i)}\left[\sqrt{\varphi(t_0,t_{i-1},t_{i+1})}\right.\right.\\
&\left.\left.-\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\sqrt{\varphi(t_0,t_{i-1},t_i)}\right]\right\}\theta\int_{-\theta\infty}^{\theta z^*-\theta\sqrt{\varphi(t_0,t_{i-1},t_{i+1})}-\theta\left[1-2\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\right]\sqrt{\varphi(t_0,t_{i-1},t_i)}}(2\pi)^{-\frac{1}{2}}
\end{aligned}$$

$$\exp\left(-\frac{1}{2}w^2\right)\Phi\left\{\left[-\theta p_2^{\theta\max(\theta r_H,\theta r_k)}(t_0)+\theta\sqrt{g(t_{i-1},t_i)+\varphi(t_0,t_{i-1},t_i)}+\theta\rho w\right]\frac{1}{\sqrt{1-\rho^2}}\right\}dw$$
$$+\theta(1+\delta r_k)P(t_0,t_{i+1})\exp\left\{2\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\ln\left[\frac{P^{-1}(t_0,t_{i-1},t_i)}{1+\delta r_H}\right]\right\}$$
$$\exp\left\{\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\sqrt{\varphi(t_0,t_{i-1},t_i)}\left[\left(1+2\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\right)\sqrt{\varphi(t_0,t_{i-1},t_i)}-2\sqrt{\varphi(t_0,t_{i-1},t_{i+1})}\right]\right\}$$
$$\theta\int_{-\theta\infty}^{\theta z^*-\theta\sqrt{\varphi(t_0,t_{i-1},t_{i+1})}+2\theta\sqrt{\frac{\hat{h}(t_{i-1},t_i)}{g(t_{i-1},t_i)}}\sqrt{\varphi(t_0,t_{i-1},t_i)}}(2\pi)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}w^2\right)$$
$$\Phi\left\{\left[-\theta p_2^{\theta\max(\theta r_H,\theta r_k)}(t_0)+\theta\rho w\right]\frac{1}{\sqrt{1-\rho^2}}\right\}dw.$$

Finally, using lemma 3, equation (82) is easily converted into the pricing solution (52). ■

F Appendix: Proof of proposition 9

Using identity (6), equation (55) can be restated as

$$\begin{aligned}&L(\theta)_{t_0}\left[r_n(t_i,t_{i+1});\theta\sup_{t_0\le s\le t_i}[\theta r_n(s,s+\delta)];t_{i+1}\right] &(83)\\ =\;&-\theta P(t_0,t_{i+1})E_{t_0}^{Q^{t_{i+1}}}\left\{\exp\left[\ln P^{-1}(t_i,t_{i+1})\right]\right\}+\theta P(t_0,t_{i+1})m(\theta)_{t_0}+\theta P(t_0,t_{i+1})\\ &E_{t_0}^{Q^{t_{i+1}}}\left\{\exp\left[\theta\sup_{t_0\le s\le t_i}\left(\theta\ln P^{-1}(s,s+\delta)\right)\right]-m(\theta)_{t_0}\middle|\sup_{t_0\le s\le t_i}\left[\theta\ln P^{-1}(s,s+\delta)\right]>\theta m(\theta)_{t_0}\right\}.\end{aligned}$$

The first expectation on the right-and-side of equation (83) is computed using the unrestricted density (15) and equation (21), while the last expectation involves the density offered by corollary 2. Therefore,

$$\begin{aligned}&L(\theta)_{t_0}\left[r_n(t_i,t_{i+1});\theta\sup_{t_0\le s\le t_i}[\theta r_n(s,s+\delta)];t_{i+1}\right] &(84)\\ =\;&-\theta P(t_0,t_{i+1})\int_{-\infty}^{+\infty}\exp(x)\,\phi\left[x;\ln P^{-1}(t_0,t_i,t_{i+1})-\frac{1}{2}g(t_0,t_i),\sqrt{g(t_0,t_i)}\right]dx+\theta P(t_0,t_{i+1})m(\theta)_{t_0}\\ &+P(t_0,t_{i+1})\int_{\theta\ln\left[m(\theta)_{t_0}\right]}^{\theta\infty}\left[\exp(y)-m(\theta)_{t_0}\right]\left\{\phi\left[y;\ln P^{-1}(t_0,t_0+\delta)-\sqrt{h(t_0,t_i)g(t_0,t_i)},\sqrt{g(t_0,t_i)}\right]\right.\\ &+2\theta\sqrt{\frac{h(t_0,t_i)}{g(t_0,t_i)}}\exp\left[2\sqrt{\frac{h(t_0,t_i)}{g(t_0,t_i)}}\ln\frac{P^{-1}(t_0,t_0+\delta)}{y}\right]\Phi\left[\theta\frac{\ln\left(\frac{P^{-1}(t_0,t_0+\delta)}{y}\right)+\sqrt{h(t_0,t_i)g(t_0,t_i)}}{\sqrt{g(t_0,t_i)}}\right]\\ &\left.+\exp\left[2\sqrt{\frac{h(t_0,t_i)}{g(t_0,t_i)}}\ln\frac{P^{-1}(t_0,t_0+\delta)}{y}\right]\phi\left[y;\ln P^{-1}(t_0,t_0+\delta)+\sqrt{h(t_0,t_i)g(t_0,t_i)},\sqrt{g(t_0,t_i)}\right]\right\}dy.\end{aligned}$$

Solving the univariate integrals contained in equation (84) -using standard calculus- it is easy to derive the explicit solution (59). ■

G Appendix: Proof of proposition 10

G.1 Case 1: $\theta r_k \ge \theta r_n(t_0,t_0+\delta)$

If $\theta r_k \ge \theta r_n(t_0,t_0+\delta)$, where $\theta = 1\,(-1)$ for a fixed strike lookback caplet (floorlet), then it is possible that the terminal payoff of the lookback option -as given by equation (54)- turns out to be zero. Therefore, the fair price of such lookback option is obtained by computing the expectation contained in equation (56) through the density (57):

$$
\begin{aligned}
& L(\theta)_{t_0}\left[\theta \sup_{t_0 \le s \le t_i}[\theta r_n(s, s+\delta)]; \theta r_k \ge \theta r_n(t_0, t_0+\delta); t_{i+1}\right] \qquad (85)\\
= \quad & P(t_0, t_{i+1}) \int_{\theta \ln(1+\delta r_k)}^{\theta \infty} [\exp(y) - (1+\delta r_k)] \left\{ \phi\left[y; \ln P^{-1}(t_0, t_0+\delta) - \sqrt{h(t_0, t_i) g(t_0, t_i)}, \sqrt{g(t_0, t_i)}\right] \right.\\
& + 2\theta \sqrt{\frac{h(t_0, t_i)}{g(t_0, t_i)}} \exp\left[2\sqrt{\frac{h(t_0, t_i)}{g(t_0, t_i)}} \ln \frac{P^{-1}(t_0, t_0+\delta)}{y}\right] \Phi\left[\theta \frac{\ln\left(\frac{P^{-1}(t_0, t_0+\delta)}{y}\right) + \sqrt{h(t_0, t_i) g(t_0, t_i)}}{\sqrt{g(t_0, t_i)}}\right]\\
& \left. + \exp\left[2\sqrt{\frac{h(t_0, t_i)}{g(t_0, t_i)}} \ln \frac{P^{-1}(t_0, t_0+\delta)}{y}\right] \phi\left[y; \ln P^{-1}(t_0, t_0+\delta) + \sqrt{h(t_0, t_i) g(t_0, t_i)}, \sqrt{g(t_0, t_i)}\right] \right\} dy.
\end{aligned}
$$

Equation (85) possess the same structure as the second term on the right-hand-side of equation (84) and can be solved in a similar fashion, yielding the explicit solution provided by formulae (60) with $\theta \max[\theta r_k, \theta r_n(t_0, t_0+\delta)]$ replaced by r_k.

G.2 Case 2: $\theta r_k < \theta r_n(t_0, t_0+\delta)$

If $\theta r_k < \theta r_n(t_0, t_0+\delta)$, then the terminal payoff of the fixed strike lookback option will be surely positive, and equation (54) can be rewritten as:

$$
\begin{aligned}
& L(\theta)_{t_{i+1}}\left[\theta \sup_{t_0 \le s \le t_i}[\theta r_n(s, s+\delta)]; \theta r_k < \theta r_n(t_0, t_0+\delta); t_{i+1}\right] \qquad (86)\\
= \quad & \left\{ \max\left(\sup_{t_0 < s \le t_i}[\theta r_n(s, s+\delta)], \theta r_n(t_0, t_0+\delta)\right) - \theta r_k \right\} \delta\\
= \quad & \delta[\theta r_n(t_0, t_0+\delta) - \theta r_k] + \delta \left\{ \sup_{t_0 < s \le t_i}[\theta r_n(s, s+\delta)] - \theta r_n(t_0, t_0+\delta) \right\}^{+}\\
= \quad & \delta[\theta r_n(t_0, t_0+\delta) - \theta r_k] + L(\theta)_{t_{i+1}}\left[\theta \sup_{t_0 \le s \le t_i}[\theta r_n(s, s+\delta)]; r_n(t_0, t_0+\delta); t_{i+1}\right],
\end{aligned}
$$

where the last equality follows from definition 6.

Assuming that measure $\mathcal{Q}^{t_{i+1}}$ exists, then

$$
\begin{aligned}
& L(\theta)_{t_0}\left[\theta \sup_{t_0 \le s \le t_i}[\theta r_n(s, s+\delta)]; \theta r_k < \theta r_n(t_0, t_0+\delta); t_{i+1}\right] \qquad (87)\\
= \quad & P(t_0, t_{i+1}) \delta[\theta r_n(t_0, t_0+\delta) - \theta r_k] + L(\theta)_{t_0}\left[\theta \sup_{t_0 \le s \le t_i}[\theta r_n(s, s+\delta)]; r_n(t_0, t_0+\delta); t_{i+1}\right],
\end{aligned}
$$

where the second term on the right-hand-side of equation (87) is computed through equation (85), but with r_k replaced by $r_n(t_0, t_0+\delta)$. It is easy to show that such term generates the explicit solution provided by formulae (60) with $\theta \max[\theta r_k, \theta r_n(t_0, t_0+\delta)]$ replaced by $r_n(t_0, t_0+\delta)$. ■

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Table 1: Pricing of a three-year up-and-in cap, under a three-factor time-homogeneous HJM model

Caplet maturity (t_i) (years)	Standard Monte Carlo (N = 200,000; S = 520; M = 260)			Approximate analytical solution		Regular cap	
	price	std. error	% std. error	price	% errors	price	% errors
0.25	0.000105	1.12E-06	1.071%	0.000116	10.866%	0.000471	349.178%
0.5	0.000348	2.02E-06	0.579%	0.000361	3.685%	0.000644	84.905%
0.75	0.000538	2.50E-06	0.465%	0.000544	1.067%	0.000761	41.501%
1	0.000677	2.82E-06	0.416%	0.000676	-0.228%	0.000849	25.358%
1.25	0.000777	3.04E-06	0.391%	0.000775	-0.291%	0.000918	18.113%
1.5	0.000856	3.21E-06	0.375%	0.000850	-0.694%	0.000972	13.470%
1.75	0.000917	3.34E-06	0.364%	0.000910	-0.796%	0.001015	10.677%
2	0.000963	3.44E-06	0.357%	0.000957	-0.666%	0.001050	8.981%
2.25	0.001004	3.51E-06	0.350%	0.000994	-0.996%	0.001077	7.296%
2.5	0.001036	3.58E-06	0.345%	0.001024	-1.237%	0.001099	6.046%
2.75	0.001059	3.62E-06	0.342%	0.001047	-1.096%	0.001116	5.432%
Cap	0.008281			0.008253	-0.337%	0.009972	20.420%
Time	> 1 day			11.7 seconds			

Caplet prices are for \$1 of Notional Value. $r_k = 6.045\%$; $r_u = 7\%$; $\delta = 0.25$ years.
N is the number of simulations run.
S is the number of discrete time-steps per year.
M is the number of barrier' monitoring time-steps per year.
% std. error is standard error divided by option price estimate.

Table 2: Pricing of a two-year down-and-out cap, under a three-factor time-homogeneous HJM model

Caplet maturity (t_i) (years)	Standard Monte Carlo (N = 200,000; S = 520; M = 260)			Approximate analytical solution		Regular cap	
	price	std. error	% std. error	price	% errors	price	% errors
0.25	0.000468	1.53E-06	0.328%	0.000471	0.725%	0.000471	0.764%
0.5	0.000638	2.09E-06	0.327%	0.000642	0.593%	0.000644	0.818%
0.75	0.000754	2.48E-06	0.329%	0.000756	0.183%	0.000761	0.907%
1	0.000834	2.76E-06	0.331%	0.000836	0.261%	0.000849	1.805%
1.25	0.000889	2.98E-06	0.335%	0.000895	0.625%	0.000918	3.202%
1.5	0.000928	3.15E-06	0.339%	0.000937	0.929%	0.000972	4.667%
1.75	0.000951	3.27E-06	0.344%	0.000967	1.776%	0.001015	6.777%
Cap	0.005463			0.005505	0.765%	0.005630	3.055%
Time	> 1 day			11.75 seconds			

Caplet prices are for \$1 of Notional Value. $r_k = 6.045\%$; $r_l = 5\%$; $\delta = 0.25$ years.
N is the number of simulations run.
S is the number of discrete time-steps per year.
M is the number of barrier' monitoring time-steps per year.
% std. error is standard error divided by option price estimate.

Table 3: Pricing of a two-year partial up-and-in cap, under a three-factor time-homogeneous HJM model

Caplet maturity (t_i) (years)	Standard Monte Carlo (N = 200,000; S = 520; M = 260)			Approximate analytical solution		Regular cap	
	price	std. error	% std. error	price	% errors	price	% errors
0.25	0.000102	1.11E-06	1.087%	0.000116	10.174%	0.000471	360.588%
0.5	0.000337	2.00E-06	0.593%	0.000362	4.136%	0.000644	90.885%
0.75	0.000510	2.50E-06	0.490%	0.000530	2.938%	0.000761	49.397%
1	0.000629	2.82E-06	0.448%	0.000649	2.450%	0.000849	35.014%
1.25	0.000720	3.05E-06	0.424%	0.000739	1.888%	0.000918	27.404%
1.5	0.000793	3.23E-06	0.408%	0.000808	1.630%	0.000972	22.610%
1.75	0.000847	3.37E-06	0.397%	0.000863	1.383%	0.001015	19.774%
Cap	0.003938			0.004068	3.282%	0.005630	42.953%
Time	> 1 day			11.09 seconds			

Caplet prices are for $1 of Notional Value. $r_k = 6.045\%$; $r_u = 7\%$; $\delta = 0.25$ years.
N is the number of simulations run.
S is the number of discrete time-steps per year.
M is the number of barrier' monitoring time-steps per year.
% std. error is standard error divided by option price estimate.

Table 4: Pricing of a two-year partial down-and-out cap, under a three-factor time-homogeneous HJM model

Caplet maturity (t_i) (years)	Standard Monte Carlo (N = 200,000; S = 520; M = 260)			Approximate analytical solution		Regular cap	
	price	std. error	% std. error	price	% errors	price	% errors
0.25	0.000164	1.14E-06	0.696%	0.000102	-38.087%	0.000471	187.341%
0.5	0.00049	2.08E-06	0.424%	0.000490	0.039%	0.000644	31.473%
0.75	0.000642	2.51E-06	0.390%	0.000653	1.714%	0.000761	18.600%
1	0.000747	2.80E-06	0.374%	0.000765	2.426%	0.000849	13.672%
1.25	0.000827	3.02E-06	0.366%	0.000850	2.784%	0.000918	11.024%
1.5	0.000892	3.19E-06	0.358%	0.000915	2.618%	0.000972	8.975%
1.75	0.00094	3.32E-06	0.353%	0.000967	2.815%	0.001015	7.921%
Cap	0.004701			0.004741	0.843%	0.005630	19.752%
Time	> 1 day			14.83 seconds			

Caplet prices are for $1 of Notional Value. $r_k = 6.045\%$; $r_l = 6\%$; $\delta = 0.25$ years.
N is the number of simulations run.
S is the number of discrete time-steps per year.
M is the number of barrier' monitoring time-steps per year.
% std. error is standard error divided by option price estimate.

Table 5: Pricing of a two-year floating strike lookback cap, under a three-factor time-homogeneous HJM model

Caplet maturity (t_i) (years)	Standard Monte Carlo (N = 200,000; S = 520; M = 260)			Approximate analytical solution	
	price	std. error	% std. error	price	% errors
0.25	0.000875	1.60E-06	0.182%	0.000943	7.723%
0.5	0.001242	2.18E-06	0.176%	0.001288	3.668%
0.75	0.001506	2.60E-06	0.172%	0.001524	1.222%
1	0.001714	2.90E-06	0.169%	0.001702	-0.712%
1.25	0.001883	3.14E-06	0.167%	0.001841	-2.204%
1.5	0.002031	3.34E-06	0.164%	0.001952	-3.884%
1.75	0.002152	3.51E-06	0.163%	0.002042	-5.121%
Cap	0.011403			0.011292	-0.975%
Time	> 1 day			11.7 seconds	

Caplet prices are for \$1 of Notional Value. $\delta = 0.25$ years.
N is the number of simulations run.
S is the number of discrete time-steps per year.
M is the number of extrema' monitoring time-steps per year.
% std. error is standard error divided by option price estimate.

Table 6: Pricing of a two-year fixed strike lookback cap, under a three-factor time-homogeneous HJM model

Caplet maturity (t_i) (years)	Standard Monte Carlo (N = 200,000; S = 520; M = 260)			Approximate analytical solution		Regular cap	
	price	std. error	% std. error	price	% errors	price	% errors
0.25	0.000876	1.60E-06	0.182%	0.000943	7.638%	0.000471	-46.190%
0.5	0.001242	2.18E-06	0.176%	0.001287	3.587%	0.000644	-48.193%
0.75	0.001505	2.58E-06	0.172%	0.001521	1.060%	0.000761	-49.418%
1	0.001713	2.89E-06	0.169%	0.001695	-1.056%	0.000849	-50.427%
1.25	0.001881	3.12E-06	0.166%	0.001829	-2.756%	0.000918	-51.222%
1.5	0.002025	3.31E-06	0.163%	0.001935	-4.452%	0.000972	-52.012%
1.75	0.002147	3.46E-06	0.161%	0.002018	-6.016%	0.001015	-52.731%
Cap	0.011390			0.011229	-1.421%	0.005630	-50.572%
Time	> 1 day			16.04 seconds			

Caplet prices are for \$1 of Notional Value. $r_k = 6.045\%$; $\delta = 0.25$ years.
N is the number of simulations run.
S is the number of discrete time-steps per year.
M is the number of extrema' monitoring time-steps per year.
% std. error is standard error divided by option price estimate.