Multi-Factor and Analytical Valuation of Treasury Bond Futures with an Embedded Quality Option[1]

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Abstract

A closed-form but approximate pricing solution is proposed for the *quality option* embedded in Treasury bond futures contracts, under a multi-factor Gaussian Heath, Jarrow and Morton (1992) framework. Using a *conditioning* approximation, in the sense of Curran (1994) and Rogers and Shi (1995), or a *rank 1* approximation, as proposed by Brace and Musiela (1994), and no matter the diversity of the underlying delivery basket or the dimension of the term structure model under analysis, it is possible to write the price of a Treasury bond future (with an embedded quality option) as a weighted average of zero-coupon bond futures prices, with weights that simply involve the univariate normal distribution function. Based on a Monte Carlo study, both approximations are shown to be extremely accurate and expedite, as well as almost indistinguishable amongst themselves.

The proposed pricing model is then applied to test the magnitude of the quality option for the EUREX' Treasury bond futures contracts, during the period between May 1999 and September 2001. For that purpose, the term structure of risk-free interest rates is estimated in accordance with the *consistent* parametrization suggested by Bjork and Christensen (1999), and using all the German Treasury bonds traded during the sample period. Then, and for each cross-section, the model' volatility function is calibrated to the market prices of all (traded) Euro-Schatz, Euro-Bobl, and Euro-Bund futures contracts, through the proposed approximate pricing solution. A remarkable fit to the EUREX market of Treasury bond futures is obtained through a simple three-factor and time-homogeneous interest rate model' specification.

The empirical analysis suggests that the quality option possesses an insignificant impact on EUREX' futures prices: on average, this delivery option only accounts for 5 basis points of the futures prices.

Key words: Gaussian HJM multi-factor models, Quality option, Consistent forward rate curves, Treasury bond futures, EUREX market.

JEL Classification: C15, E43, G13.

1 Introduction

Treasury bond futures contracts contain a variety of features -known as *delivery options*- that provide the party with a short position some flexibility concerning the delivery process. Such features allow the futures' seller to decide what (deliverable) bond to deliver and/or to choose when such delivery occurs. Hence, two types of delivery options are usually considered in the literature: the *quality option* and the *timing option*.

The quality option is simply the contract feature that allows the short position to deliver, on the delivery day, any of the deliverable bonds specified by the exchange. For instance, the deliverable basket of the Euro-Bund futures contracts, traded on the EUREX, includes (on the delivery day) all the German Treasury bonds with a remaining time-to-maturity between 8.5 and 10.5 years, while for the US Treasury bond futures contracts, traded on the Chicago Board of Trade (CBOT), the short can deliver any US government bond with at least 15 years to maturity or to first call. For each deliverable bond (of each delivery month) the exchange defines, *a priori*, a (possibly different) conversion factor that will adjust the invoice amount to be paid by the futures' buyer, with the purpose of establishing the indifference of the long position towards the choice of the deliverable issue. However, such conversion factors are computed as being the (unit face value) clean prices of the deliverable bonds, on the delivery day, such that the yields-to-maturity of all deliverable issues are equal (to the notional' coupon rate). Therefore, the conversion factors' system is only able to define, not an exact[1] but, an approximate equivalence relation amongst the different deliverable bonds and, consequently, the quality option value may not vanish.

The timing option arises whenever the futures' seller is allowed to select the moment (during the delivery month) where delivery takes place, and can be decomposed into three sub-categories: the *accrued interest*, the *wild card* and the *end-of-the-month* options. The accrued interest option exists for those futures contracts where the short position is able to make delivery on any trading day of the maturity month.[2] This is the case for the CBOT' futures contracts but not for the EUREX' bond futures, since for the latter delivery must take place on the tenth calendar day of each (quarterly) delivery month. The wild card option is present whenever there is a time-gap, on each trading day, between the ending of the futures trading (i.e. the fixing of the futures' settlement price) and the deadline for delivery announcement, as long as the cash market is open during that period. For example, the seller of a CBOT' futures contract can trade on the cash market (until 4 p.m. - Chicago time) and postpone the delivery announcement between 2 p.m. -ending of futures' trading- and 8 p.m., based on an already known futures' settlement price. Finally, the end-of-the-month option arises whenever the delivery announcement can be postponed even after the last trading day for futures contracts. For instance, because all CBOT' futures contracts that are still open on the last trading day (i.e. on the eighth-to-last business day of the delivery month) can be delivered on any of the following seven business days, based on the futures' settlement price defined on the last trading day, then the short position can optimize the delivery decision through the spot market activity on the same time period.

Although intended to broaden the scope of the futures contracts to a wider clientele and to avoid liquidity restrictions on the corresponding spot market, such delivery options increase the pricing (and hedging) complexity of these interest rate derivatives. In fact, the futures' seller, who benefits from those delivery options, must be charged for their fair value, not explicitly but rather,

[1] Unless the implausible scenario of a flat yield curve (at the level of the theoretical issue' coupon rate) is observed on the delivery day. In this case, the quality option would be worthless to the short.

[2] The "accrued interest" denomination arises from the fact that postponing the delivery decision is only rational as long as the daily coupon accruing from a long position on a deliverable bond is high enough to compensate the "cost-of-carrying" such long position.

through a "bid-down" adjustment of the market futures' price, which is also meant to compensate the buyer for the corresponding additional "delivery risk".

The measurement and modelling of this (negative) impact of the delivery options on the market price of futures contracts has been extensively studied in the literature, through different methods and with disparate empirical results. Using a multiple regression analysis on (deliverable) bonds' holding returns, Benninga and Smirlock (1985) provided statistical evidence of a significative impact of the quality option on CBOT' T-bond futures prices. Similarly but through a Monte Carlo study, Kane and Marcus (1986) also found a significant value for the quality option (between 1.39% and 4.6% of the futures' contract size, three months before delivery). By opposition, Livingston (1987), Hegde (1988), Barnhill and Seale (1988) and Barnhill (1990) have all found insignificant estimates for the quality option value, through different methodologies based on the "cost-of-carry" model.[3]

Concerning the modelling of the quality option value, the initial valuation approach consisted in assuming a stochastic process directly on bond prices. Gay and Manaster (1984) applied the Margrabe (1978) exchange option formulae to the valuation of the quality option implicit in CBOT' wheat futures contracts (with only two deliverable assets), and have found a significant impact on futures market prices (of about 2.2% of contract size, on average). Boyle (1989) approximated the value of the quality option on any finite number of deliverable assets, under the assumption of joint lognormality, and also found significant quality option' estimates. On the contrary, Hemler (1990) estimated an average value for the quality option embedded in a three-month T-bond contract lower than 0.3% (of contract size), while still adopting a joint lognormal probability distribution for all the bonds contained in the deliverable set (which he has assumed to be restricted to the current three cheapest-to-deliver bonds). However, it is well known that the assumption of a joint lognormal distribution process is not only inappropriate for valuing interest rate contingent claims but also computationally expensive since the covariance structure amongst all bond returns would have to be estimated. Moreover, while the quality option value should mainly depend on the variability of the term structure of interest rates, all the previously cited modelling approaches wrongly assume a deterministic interest rate setting where the marking-to-market practice is ignored and, therefore, futures contracts are essentially valued as if they were forward contracts.

More recent studies about the significance of the quality option have taken into account the stochastic nature of the term structure of interest rates, by considering an equilibrium framework where all bond prices are generated through the dynamics of a small number of state variables or through an arbitrage-free model where all interest rate contingent claims are priced consistently with an initially observed spot yield curve. The former approach -based on a framework such as Vasicek (1977) or Cox, Ingersoll and Ross (1985b)- includes, among others, Chen (1997), Carr and Chen (1997) and Bick (1997). The latter uses the Heath et al. (1992) (HJM, hereafter) setting and includes, for instance, the works of Ritchken and Sankarasubramanian (1992), Ritchken and Sankarasubramanian (1995), and Lin and Paxson (1995).

Carr and Chen (1997) have obtained exact and quasi closed-form solutions for the quality option under a one- and a two-factor Cox et al. (1985b) framework. They applied their solutions to the CBOT' T-bond futures contracts (from Jan/87 to Dec/91) and concluded that the magnitude of the quality option is potentially large. Chen (1997) derives upper bounds for the delivery options embedded in Treasury bond futures prices and tests them for the same time-period, using a two-factor Cox et al. (1985b) term structure model. He concludes that delivery options can significantly affect futures prices, being the main impact derived from the quality option presence. Under a Vasicek (1977) framework but for futures contracts on Treasury zero-coupon bonds, Bick (1997)

[3] The "cost-of-carry" model treats futures as forward contracts, since it assumes that the futures price equals the underlying spot price (of the cheapest-to-deliver bond) minus the unobtainable "carry return" (coupons) and plus the avoided "carry cost" (i.e. the cost of funding the purchase of the underlying).

has also derived an analytical futures pricing formula.

Ritchken and Sankarasubramanian (1992) have considered a one-factor HJM term structure model with a Gaussian -Vasicek (1977) type- volatility structure. They provided an exact closed-form solution for futures prices with an embedded quality option but on pure discount bonds, and priced numerically futures contracts on coupon-bearing bonds. Later, Ritchken and Sankarasubramanian (1995) extended their initial single-factor model towards a two-factor Gaussian HJM framework, where the quality option is valued numerically. Using CBOT' T-bond futures contracts and different values for the model' volatility parameters, they have shown that the quality option value can be significative (and larger than the one impled by a single-factor model): on average (and three months prior to the delivery day), it is found to be equal to 0.1%, 1% and 2% for two-, ten- and fifteen-year futures contracts, respectively. Lin and Paxson (1995) implement a one-factor Gaussian HJM model, through a discrete-time binomial grid, for pricing the German government bond (bund) futures contracts traded on the London International Financial Futures and Options Exchange (LIFFE), from Dec/88 to Nov/91. Their results show that the conventional quality option value implied in bund futures contracts is small, with an average value, three months prior to delivery, equal to only 9 basis points, while what they call the *new issue option*[4] is only found to be worth about 9.7 basis points.

The present paper proposes, for the first time to the authors' knowledge, an analytical valuation formula for the quality option embedded in Treasury bond futures contracts, under a multi-factor Gaussian HJM term structure model and can, therefore, be understood as an extension to Ritchken and Sankarasubramanian (1992) or Ritchken and Sankarasubramanian (1995). Such closed-form pricing solution is shown to be obtainable from two alternative approximations already proposed in the literature, although in different contexts: the *conditioning approach*, initiated by Curran (1994) and Rogers and Shi (1995) in the context of Asian option pricing; or, the *rank 1 approximation*, previously used by Brace and Musiela (1994) for the pricing of European swaptions. For any of these two approximations, the value of a Treasury bond futures contract (with an embedded quality option) will be written as a weighted average of zero-coupon bond futures prices, where the weights are expressed in terms of the univariate normal distribution function.[5]

As the previous bibliographic references highlight, the empirical evidence concerning the magnitude of the quality option is not consensual. In this paper we will use the previously mentioned analytical pricing formulae to test the relevance of the quality option on the EUREX market (from May/99 to Sep/01). The EUREX derivatives market was chosen in order to better isolate the impact of the quality option on futures prices, because the existence of a uniquely admissible delivery day ensures that no timing options would need to be considered. Similarly to Lin and Paxson (1995), the quality option value will be shown to be of small magnitude for the German Treasury bond futures contracts under analysis.

Next sections are organized as follows. Section 2 establishes the fundamental no-arbitrage restrictions -valid for any term structure model- that enable the quality option component to be detached, for valuation purposes, from the futures price. Based on the multi-factor Gaussian HJM model presented in section 3, section 4 prices Treasury bond futures contracts without any delivery options. Then, section 5 presents the main theoretical contribution of the paper, that is the analytical pricing solution for Treasury bond futures contracts with an embedded quality option, and tests its numerical accuracy through a Monte Carlo study. Section 6 proposes a parametric family of forward rate curves that will allow the spot yield curve to be estimated

[4] That is the additional quality option value that arises whenever the exchange allows new bond issues to be included in the deliverable set between the valuation date and the final delivery day.

[5] Carr and Chen (1997) have obtained a similar representation involving the non-central chi-square distribution function, but under a one-factor Cox et al. (1985b) model.

consistently -in the sense of Bjork and Christensen (1999)- with the HJM model under analysis, whose dimension is also tested through a principal components analysis. Finally, section 7 is devoted to the empirical analysis of the quality option relevance for the Treasury bond futures contracts traded on the EUREX market. Section 8 summarizes the main conclusions, while all accessory proves are relegated to the appendix.

2 Model-independent valuation of the quality option

Next propositions present some general and well known arbitrage pricing restrictions on the value of the quality option, which do not depend on any specific interest rate modelling assumptions. The notation is borrowed from Bühler, Düllmann and Windfuhr (2001, page 30).

Hereafter, uncertainty will be represented by a filtered probability space $(\Omega, \mathcal{F}, \mathcal{Q}, \mathbb{F})$ satisfying the usual technical conditions, and where $\mathcal{Q}$ is the risk-neutral measure obtained when the *money-market account* is taken to be the numeraire of the underlying continuous-time economy.[6] Furthermore,

Assumption 1 Futures contracts are assumed to be continuously marked-to-market. That is increases in the futures price are continuously (instead of daily) credited to the margin account of the holder of a long position and debited to the futures' seller.

Assumption 2 There are no *timing options.*

Remark 1 *This is the case for the Treasury bond futures contracts traded on EUREX, which will be used in the forthcoming empirical analysis. Moreover, because of the complex nature of the delivery options, it is common practice to focus the analysis on "one option at a time".*

Next proposition provides a general valuation formula for Treasury bond futures with an embedded quality option.

Proposition 1 *The time-t fair price of a bond futures contract that matures at time T_f $(\geq t)$ and is written on a delivery basket containing m deliverable Treasury coupon-bearing bonds is equal to:*

$$H(t, T_f, \{1, \ldots, m\}) = \mathbb{E}_{\mathcal{Q}}\left\{ \min_{j=1}^{m} \left[\frac{CB_j(T_f)}{cf_j} \right] \middle| \mathcal{F}_t \right\}, \tag{1}$$

where $CB_j(t)$ represents the time-t clean price of the j^{th} deliverable bond and cf_j is the corresponding conversion factor.

Proof. Following, for instance, Chen (1997), it is well known that the existence of a quality option enables the short position to choose at time T_f the cheapest-to-deliver bond. Therefore, the futures seller' time-T_f payoff is equal to:

$$\max_{j=1}^{m} \left\{ [cf_j H(T_f, T_f, \{1, \ldots, m\}) + AI_j(T_f)] - [CB_j(T_f) + AI_j(T_f)] \right\}, \tag{2}$$

where $AI_j(T_f)$ denotes the j^{th} deliverable bond time-T_f accrued interest. However, in order to preclude arbitrage opportunities (and neglecting the existence of transaction costs), the payoff (2) must be identically zero -see, for example, Duffie (1989, page 327). Hence, solving for the terminal futures price, yields:

[6] That is, discounted spot price processes are assumed to be martingales under measure $\mathcal{Q}$.

$$H\left(T_f, T_f, \{1, \ldots, m\}\right) = \min_{j=1}^{m} \left[\frac{CB_j\left(T_f\right)}{cf_j}\right]. \quad (3)$$

Since a futures price is simply the expectation of the spot price on the delivery date -see, for instance, Cox, Ingersoll and Ross (1981, equation 46)- then equation (1) follows. ■

Ignoring the quality option value is equivalent to assume that the choice of the bond issue to be delivered at time T_f is made irreversibly on the valuation date (time t). Definition 2 assumes, similarly, that the choice of the cheapest-to-deliver bond is made as if it could not be postponed until the delivery date.

Definition 2 *The time-t cheapest-to-deliver bond is the deliverable bond $j^* \in \{1, \ldots, m\}$ such that:*

$$j^* = \arg\min_{j=1}^{m} \left\{\mathbb{E}_{\mathcal{Q}}\left[\left.\frac{CB_j\left(T_f\right)}{cf_j}\right| \mathcal{F}_t\right]\right\}. \quad (4)$$

Remark 2 *Bond j^* is the deliverable bond that, at time t, the short would elect to deliver, at time T_f, if there was no quality option embedded in the futures contract.*

Proposition 3 *The time-t fair price of a bond futures contract that matures at time T_f $(\geq t)$ and is written on the specific deliverable bond j^* is equal to*

$$H\left(t, T_f, \{j^*\}\right) = \min_{j=1}^{m} \left\{\mathbb{E}_{\mathcal{Q}}\left[\left.\frac{CB_j\left(T_f\right)}{cf_j}\right| \mathcal{F}_t\right]\right\}. \quad (5)$$

Remark 3 *This is the fair value that a Treasury bond futures contract should possess if there was no quality option.*

Proof. The time-T_f payoff for the futures' seller, which must be zero in an arbitrage-free market, is:

$$\left[cf_{j^*} H\left(T_f, T_f, \{j^*\}\right) + AI_{j^*}\left(T_f\right)\right] - \left[CB_{j^*}\left(T_f\right) + AI_{j^*}\left(T_f\right)\right] = 0. \quad (6)$$

Solving for the future price,

$$H\left(T_f, T_f, \{j^*\}\right) = \frac{CB_{j^*}\left(T_f\right)}{cf_{j^*}}, \quad (7)$$

applying expectations, and using identity (4), the pricing formula (5) arises. ■

Finally, the value of the quality option can be obtained by comparing both Treasury bond futures prices: without and with the possibility of postponing the choice of the cheapest-to-deliver.

Definition 4 *The time-t fair value of the embedded quality option is equal to:*

$$QO\left(t, T_f, \{1, \ldots, m\}\right) = H\left(t, T_f, \{j^*\}\right) - H\left(t, T_f, \{1, \ldots, m\}\right). \quad (8)$$

Remark 4 *This is the value that results from the fact that the short can postpone the election of the bond to be delivered from time t to time T_f. From equation (8), the value of the quality option can be defined as the difference between the price of a futures contract allowing only one bond to be delivered (the current cheapest-to-deliver), and the price of a similar futures contract which allows several bonds to be delivered.*

3 Multi-factor Gaussian HJM model

Treasury bond futures contracts (with an embedded quality option) will be valued using a multi-factor Heath et al. (1992) Gaussian term structure model. The choice of the HJM framework is necessary in order to obtain a perfect fit to the market prices of all deliverable Treasury bonds, that is in order to incorporate into the model the observable prices of the primary assets for the derivative (futures contract) whose value is intended to be modelled. The model' volatility structure will be restricted to be deterministic -Gaussian assumption- for analytical purposes. Because, for each cross section, the model' parameters will be estimated by minimizing the absolute (percentage) deviations between model and market bond futures' prices, it is essential to be able to derive a (*fast*) closed-form solution for the quality option included in such market quotes. Such analytical pricing formulae will be constructed in the context of a multi-factor model in order to enhance the model' fit to the Treasury bond futures market and to accommodate the Principal Components Analysis' (PCA, henceforth) usual prescription of three stylized factors: *level*, *slope* and *curvature* (see, for instance, Litterman and Scheinkman (1991)). Such PCA recommendation will be empirically tested in subsection 6.3.

The Gaussian HJM model under use can be formulated in terms of risk-free pure discount bond prices, which are assumed to evolve through time (under measure $\mathcal{Q}$) according to the following stochastic differential equation:

$$\frac{dP(t,T)}{P(t,T)} = r(t)\,dt + \underline{\sigma}(t,T)' \cdot d\underline{W}^{\mathcal{Q}}(t)\,, \tag{9}$$

where $P(t,T)$ represents the time-t price of a (unit face value and default-free) zero coupon bond expiring at time T $(\leq t)$, $r(t)$ is the time-t instantaneous spot rate, $\cdot$ denotes the inner product in $\Re^n$, and $\underline{W}^{\mathcal{Q}}(t) \in \Re^n$ is a standard Brownian motion, initialized at zero and generating the augmented, right continuous and complete filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq t_0\}$, where t_0 denotes the current time.

The n-dimensional adapted volatility function, $\underline{\sigma}(\cdot, T) : [t_0, T] \to \Re^n$, is assumed to satisfy the usual mild measurability and integrability requirements -as stated, for instance, in Lamberton and Lapeyre (1996, theorem 3.5.5)- as well as the boundary condition[7] $\underline{\sigma}(u,u) = \underline{0} \in \Re^n, \forall u \in [t_0, T]$. Moreover, for reasons of analytical tractability, that is, in order to obtain lognormally distributed pure discount bond prices,

Assumption 3 The volatility function $\underline{\sigma}(\cdot,\cdot)$ is assumed to be deterministic.

Remark 5 *Nevertheless, the proposed multi-factor Gaussian HJM model is not necessarily Markovian or time-homogeneous.*

Remark 6 *As shown by Pang (1998, subsection 1.1.2), the proposed framework (and, therefore, all the forthcoming analytical pricing solutions) can be easily generalized towards a Gaussian random field term structure model, in the sense of Kennedy (1994).*

4 Bond futures contracts without delivery options

The purpose of this section is to specialize equation (5) to the context of the interest rate model under analysis. Although such task has already been accomplished under different setups (see, for instance, El Karoui, Lepage, Myneni, Roseau and Viswanathan (1991, equations 47 and 48)), it will be now summarized for completeness and notational consistency reasons.

[7] Consistently with the "pull-to-par" phenomena.

4.1 Forward pure discount bond prices

Let

$$P(t, T_f, T) := \frac{P(t, T)}{P(t, T_f)}, \qquad t \leq T_f \leq T, \tag{10}$$

represent the time-t forward price, for delivery at time T_f, of a zero coupon bond with maturity at time T. The symbol $:=$ means equal by definition.

Next proposition shows that such forward price is lognormally distributed, under the dynamics implied by equation (9) and subject to the deterministic volatility assumption.

Proposition 5 *Under the Gaussian HJM model (9), the time-t ($\geq t_0$) forward price for delivery at time T_f ($\geq t$), of a risk-free pure discount bond with maturity at time T ($\geq T_f$) is*

$$\begin{aligned} P(t, T_f, T) = \quad & P(t_0, T_f, T) \exp\left\{ -\frac{1}{2} \int_{t_0}^{t} \left[\left\| \underline{\sigma}(s, T) \right\|^2 - \left\| \underline{\sigma}(s, T_f) \right\|^2 \right] ds \right. \\ & \left. + \int_{t_0}^{t} \left[\underline{\sigma}(s, T) - \underline{\sigma}(s, T_f) \right]' \cdot d\underline{W}^{Q}(s) \right\}, \end{aligned} \tag{11}$$

where $\|\cdot\|$ denotes the Euclidean norm in $\Re^n$.

Proof. Using equation (9) and applying Itô's lemma to $\ln P(t, T)$,

$$d \ln P(t, T) = \left[r(t) - \frac{1}{2} \left\| \underline{\sigma}(t, T) \right\|^2 \right] dt + \underline{\sigma}(t, T)' \cdot d\underline{W}^{Q}(t). \tag{12}$$

Integrating both terms of the previous stochastic differential equation over the time-interval $[t_0, t]$,

$$\ln P(t, T) = \ln P(t_0, T) + \int_{t_0}^{t} \left[r(s) - \frac{1}{2} \left\| \underline{\sigma}(s, T) \right\|^2 \right] ds + \int_{t_0}^{t} \underline{\sigma}(s, T)' \cdot d\underline{W}^{Q}(s). \tag{13}$$

Subtracting $\ln P(t, T_f)$ from both sides of equation (13) while also applying equation (13) to $\ln P(t, T_f)$,

$$\begin{aligned} \ln \frac{P(t, T)}{P(t, T_f)} = \quad & \ln \frac{P(t_0, T)}{P(t_0, T_f)} - \frac{1}{2} \int_{t_0}^{t} \left[\left\| \underline{\sigma}(s, T) \right\|^2 - \left\| \underline{\sigma}(s, T_f) \right\|^2 \right] ds \\ & + \int_{t_0}^{t} \left[\underline{\sigma}(s, T) - \underline{\sigma}(s, T_f) \right]' \cdot d\underline{W}^{Q}(s). \end{aligned} \tag{14}$$

Finally, combining equation (14) with definition (10), proposition 5 follows. ■

4.2 Futures on default-free pure discount bonds

Next proposition offers an analytical pricing solution for a zero-coupon bond futures contract, which essentially differs from the previous forward contract because of the continuous marking-to-market assumption.

Proposition 6 *Under the HJM model (9), the time-t_0 price, $F(t_0, T_f, T)$, of a futures contract for delivery at time T_f ($\geq t_0$) and on a default-free zero-coupon bond with expiry date at time T ($\geq T_f$) is equal to*

$$F(t_0, T_f, T) = P(t_0, T_f, T) \exp\left[-\frac{1}{2} \eta(t_0, T_f, T) + \frac{1}{2} \varphi(t_0, T_f, T) \right], \tag{15}$$

where

$$\eta(t_0, T_f, T) := \int_{t_0}^{T_f} \left[\|\underline{\sigma}(s, T)\|^2 - \|\underline{\sigma}(s, T_f)\|^2 \right] ds, \tag{16}$$

and

$$\varphi(t_0, T_f, T) := \int_{t_0}^{T_f} \|\underline{\sigma}(s, T) - \underline{\sigma}(s, T_f)\|^2 ds. \tag{17}$$

Proof. Since a futures price is simply the expectation of the spot price on the delivery date, then

$$F(t_0, T_f, T) = \mathbb{E}_{\mathcal{Q}}\left[P(T_f, T) | \mathcal{F}_{t_0} \right]. \tag{18}$$

Using proposition 5,

$$\begin{aligned} F(t_0, T_f, T) &= P(t_0, T_f, T) \exp\left\{ -\frac{1}{2} \int_{t_0}^{T_f} \left[\|\underline{\sigma}(s, T)\|^2 - \|\underline{\sigma}(s, T_f)\|^2 \right] ds \right\} \\ & \mathbb{E}_{\mathcal{Q}}\left\{ \exp\left[\int_{t_0}^{T_f} (\underline{\sigma}(s, T) - \underline{\sigma}(s, T_f))' \cdot d\underline{W}^{\mathcal{Q}}(s) \right] \middle| \mathcal{F}_{t_0} \right\}. \end{aligned} \tag{19}$$

The expected value appearing on the right-hand side of equation (19) is a moment generating function, with a coefficient equal to +1, of the random variable $\int_{t_0}^{T_f} [\underline{\sigma}(s, T) - \underline{\sigma}(s, T_f)]' \cdot d\underline{W}^{\mathcal{Q}}(s)$. Considering, for instance, Arnold (1992, corollary 4.5.6), it follows that[8]

$$\int_{t_0}^{T_f} [\underline{\sigma}(s, T) - \underline{\sigma}(s, T_f)]' \cdot d\underline{W}^{\mathcal{Q}}(s) \backsim N^1(0, \varphi(t_0, T_f, T)), \tag{20}$$

and equation (15) is obtained. ■

Remark 7 *As noticed by El Karoui et al. (1991, page 13), futures prices depend not only on the initial term structure of interest rates - as it is the case for forward contracts -but also on the volatility function. Hence, and after parameterizing function $\underline{\sigma}(\cdot, T) : [t_0, T] \to \Re^n$, it will be possible to estimate the HJM model' parameters by fitting, cross-sectionally, market quotes of interest rate futures contracts.*

4.3 Futures on default-free coupon-bearing bonds

It is now possible to offer a (model' dependent) closed-form solution for a Treasury bond futures contract without any delivery option.

Proposition 7 *Under the HJM model (9), the time-t_0 fair price, $H(t_0, T_f, \{j^*\})$, of a bond futures contract that matures at time T_f $(\geq t_0)$, is written on a delivery basket containing m deliverable Treasury coupon-bearing bonds but that possesses no delivery options is equal to*

$$H(t_0, T_f, \{j^*\}) = \min_{j=1}^{m} \left[-\frac{AI_j(T_f)}{cf_j} + \sum_{i=1}^{N^j} F\left(t_0, T_f, T_i^j\right) \frac{k_i^j}{cf_j} \right], \tag{21}$$

where N^j is the number of cash flows k_i^j $(i = 1, \ldots, N^j)$ paid by the j^{th} underlying coupon-bearing bond at times T_i^j $(> T_f)$.[9]

[8] Hereafter, the notation $X \backsim N^1(\mu, \sigma^2)$ is intended to mean that the one-dimensional random variable X is normally distributed, with mean μ and variance σ^2.

[9] That is, from the futures' expiry date and until the bond' maturity date.

Proof. Because

$$CB_j(T_f) = \sum_{i=1}^{N^j} P\left(T_f, T_i^j\right) k_i^j - AI_j(T_f), \tag{22}$$

then:

$$\mathbb{E}_{\mathcal{Q}}\left[\left.\frac{CB_j(T_f)}{cf_j}\right| \mathcal{F}_{t_0}\right] = -\frac{AI_j(T_f)}{cf_j} + \sum_{i=1}^{N^j} \mathbb{E}_{\mathcal{Q}}\left[\left. P\left(T_f, T_i^j\right)\right| \mathcal{F}_{t_0}\right] \frac{k_i^j}{cf_j}. \tag{23}$$

Using equation (18) and proposition 3, result (21) follows immediately. ■

5 Treasury bond futures with an embedded quality option

In order to compute the expectation contained in equation (1) it is necessary to obtain the transition probability density function, under the $\mathcal{Q}$ martingale measure, for all the stochastic factors underlying the terminal (time-T_f) deliverable coupon-bearing bond prices. From proposition 5 it follows that the number of such stochastic variables -last term in the exponential on the right-hand-side of equation (11)- increases with the dimension of vector $\underline{W}^{\mathcal{Q}}(t)$ and with the number of cash-flow payment dates generated by the deliverable basket under consideration.

The first difficulty -number of Brownian motions under consideration- could be overcome by using a simpler one-factor model. Unfortunately, the cost of such simplification would certainly be a much poor fit to the observable futures prices. The second problem -dependency on the cash flow structure of the delivery basket- could also be solved by using not an arbitrage-free term structure model but a simpler factor-model. Such approach has been successfully implemented, for instance, by Carr and Chen (1997), who price the quality option using a two-factor Cox et al. (1985b) model and the lattice methodology suggested by Longstaff and Schwartz (1992). However, the adoption of a general equilibrium term structure model would not guarantee the model fit to the market spot prices of all the deliverable bonds.

Alternatively, this paper adopts the arbitrage-free and multi-factor model defined by equation (9), and proposes two different approximations in order to reduce the dimensionality of the integration problem implicit in equation (1): one based on the *conditioning approach* initiated by Curran (1994), Rogers and Shi (1995) and Nielsen and Sandmann (2003) in the context of Asian option pricing, and extended by Nielsen and Sandmann (2002) to a stochastic interest rate setting[10]; the other one related to the *proportionality (or rank 1) assumption* used by El Karoui and Rochet (1989) and Brace and Musiela (1994) for the pricing of European options on coupon-bearing bonds.

5.1 Conditioning approach

5.1.1 Upper bound

Following, for instance, Rogers and Shi (1995), let $Z \in \Re$ be any $\mathcal{F}_{T_f}$-measurable random variable. From the law of iterative expectations and using Jensen's inequality, equation (1) can be rewritten as

$$\begin{aligned} H(t_0, T_f, \{1, \ldots, m\}) &= \mathbb{E}_{\mathcal{Q}}\left\{\left.\mathbb{E}_{\mathcal{Q}}\left[\left.\min_{j=1}^{m}\left(\frac{CB_j(T_f)}{cf_j}\right)\right| Z\right]\right| \mathcal{F}_{t_0}\right\} \\ &\leq H^u(t_0, T_f, \{1, \ldots, m\}), \end{aligned} \tag{24}$$

[10] The authors wish to thank Klaus Sandmann for suggesting us this approach.

where

$$H^u\left(t_0, T_f, \{1, \ldots, m\}\right) := \mathbb{E}_{\mathcal{Q}}\left\{\min_{j=1}^{m}\left[\mathbb{E}_{\mathcal{Q}}\left(\left.\frac{CB_j\left(T_f\right)}{cf_j}\right| Z\right)\right]\right| \mathcal{F}_{t_0}\right\} \tag{25}$$

defines an upper bound for the true futures price. Next proposition provides an explicit solution for the conditional expectation contained on the right-hand-side of equation (25), by assuming a standard normal distribution for the conditioning variable.

Proposition 8 *Under the HJM model (9), the time-t_0 fair price, $H\left(t_0, T_f, \{1, \ldots, m\}\right)$, of a bond futures contract that matures at time T_f $(\geq t_0)$ and is written on a delivery basket containing m deliverable Treasury coupon-bearing bonds is bounded from above by*

$$\begin{aligned} & H^u\left(t_0, T_f, \{1, \ldots, m\}\right) \\ = \quad & \mathbb{E}_{\mathcal{Q}}\left\{\min_{j=1}^{m}\left[-\frac{AI_j\left(T_f\right)}{cf_j}+\sum_{i=1}^{N^j} \frac{k_i^j P\left(t_0, T_f, T_i^j\right) \exp\left(-\frac{\tilde{\eta}\left(t_0, T_f, T_i^j\right)}{2}+\tilde{\varphi}\left(t_0, T_f, T_i^j\right) Z\right)}{cf_j}\right]\right| \mathcal{F}_{t_0}\right\}, \end{aligned} \tag{26}$$

where cf_j is the conversion factor of the j^{th} deliverable bond, N^j is the number of cash flows k_i^j $(i = 1, \ldots, N^j)$ paid by the such coupon-bearing bond at times T_i^j $(> T_f)$, $AI_j\left(T_f\right)$ denotes its time-T_f accrued interest, $Z \backsim N^1(0,1)$,

$$\tilde{\varphi}\left(t_0, T_f, T_i^j\right) := \mathbb{E}_{\mathcal{Q}}\left\{ Z \int_{t_0}^{T_f}\left[\underline{\sigma}\left(s, T_i^j\right)-\underline{\sigma}\left(s, T_f\right)\right]' \cdot d\underline{W}^{\mathcal{Q}}(s)\right| \mathcal{F}_{t_0}\right\}, \tag{27}$$

and

$$\tilde{\eta}\left(t_0, T_f, T_i^j\right) := \eta\left(t_0, T_f, T_i^j\right)-\varphi\left(t_0, T_f, T_i^j\right)+\tilde{\varphi}\left(t_0, T_f, T_i^j\right)^2. \tag{28}$$

Proof. Using equation (22),

$$\mathbb{E}_{\mathcal{Q}}\left[\left.\frac{CB_j\left(T_f\right)}{cf_j}\right| Z\right]=-\frac{AI_j\left(T_f\right)}{cf_j}+\sum_{i=1}^{N^j} \frac{k_i^j}{cf_j} \mathbb{E}_{\mathcal{Q}}\left[\left. P\left(T_f, T_i^j\right)\right| Z\right]. \tag{29}$$

Since $P\left(T_f, T_i^j\right)=P\left(T_f, T_f, T_i^j\right)$, proposition 5 yields:

$$\begin{aligned} \mathbb{E}_{\mathcal{Q}}\left[\left. P\left(T_f, T_i^j\right)\right| Z\right] = \; & P\left(t_0, T_f, T_i^j\right) \exp\left[-\frac{1}{2} \eta\left(t_0, T_f, T_i^j\right)\right] \\ & \mathbb{E}_{\mathcal{Q}}\left\{\exp\left[\int_{t_0}^{T_f}\left(\underline{\sigma}\left(s, T_i^j\right)-\underline{\sigma}\left(s, T_f\right)\right)' \cdot d\underline{W}^{\mathcal{Q}}(s)\right]\right| Z\right\}. \end{aligned} \tag{30}$$

Assuming that Z possesses a standard univariate normal distribution, since

$$\int_{t_0}^{T_f}\left[\underline{\sigma}\left(s, T_i^j\right)-\underline{\sigma}\left(s, T_f\right)\right]' \cdot d\underline{W}^{\mathcal{Q}}(s) \backsim N^1\left(0, \varphi\left(t_0, T_f, T_i^j\right)\right),$$

and following, for instance, Mood, Graybill and Boes (1974, page 167), then

$$\int_{t_0}^{T_f}\left[\underline{\sigma}\left(s, T_i^j\right)-\underline{\sigma}\left(s, T_f\right)\right]' \cdot d\underline{W}^{\mathcal{Q}}(s)\right| Z \backsim N^1\left(\tilde{\varphi}\left(t_0, T_f, T_i^j\right) Z, \varphi\left(t_0, T_f, T_i^j\right)-\tilde{\varphi}\left(t_0, T_f, T_i^j\right)^2\right), \tag{31}$$

where the deterministic function $\tilde{\varphi}\left(t_0, T_f, T_i^j\right)$ is defined by the covariance (27).

Applying result (31) and attending to the definition of the moment generating function of a normal random variable, equation (30) becomes

$$\begin{aligned} \mathbb{E}_{\mathcal{Q}}\left[P\left(T_f, T_i^j\right)\Big| Z\right] &= P\left(t_0, T_f, T_i^j\right)\exp\left[-\frac{1}{2}\eta\left(t_0, T_f, T_i^j\right)\right] \\ &\quad \exp\left\{\tilde{\varphi}\left(t_0, T_f, T_i^j\right)Z + \frac{1}{2}\left[\varphi\left(t_0, T_f, T_i^j\right) - \tilde{\varphi}\left(t_0, T_f, T_i^j\right)^2\right]\right\}. \end{aligned} \tag{32}$$

Combining equations (25), (29) and (32), the quasi-analytical[11] solution (26) follows for the upper bound of the futures price. ■

Remark 8 *No matter the number of Brownian motions or the number of cash-flow payment dates generated by the deliverable basket under consideration, proposition 8 reduces the valuation of the futures contract to a univariate integration problem.*

In order to obtain an analytical solution for equation (26), the following notation will be useful.

Definition 9 *Denote by z_k^*, $k = 1, \ldots, r$, all possible solutions in z for the following set of non-linear equations:*

$$\frac{CB_j\left(T_f; z\right)}{cf_j} = \frac{CB_l\left(T_f; z\right)}{cf_l}, \quad j = 1, \ldots, m-1, \quad l = j+1, \ldots, m,$$

with

$$\frac{CB_j\left(T_f; z\right)}{cf_j} := -\frac{AI_j\left(T_f\right)}{cf_j} + \sum_{i=1}^{N^j} \frac{k_i^j P\left(t_0, T_f, T_i^j\right)}{cf_j}\exp\left[-\frac{\tilde{\eta}\left(t_0, T_f, T_i^j\right)}{2} + \tilde{\varphi}\left(t_0, T_f, T_i^j\right)z\right].$$

Furthermore, set $z_0^* = -\infty$ and $z_{r+1}^* = \infty$, while assuming that all roots are arranged in increasing order, i.e. $z_1^* < z_2^* < \ldots < z_r^*$.

Next theorem contains the main contribution of this paper, namely: an approximate analytical pricing solution for Treasury bond futures contracts with embedded quality options, in the context of a multi-factor HJM Gaussian model.

Theorem 10 *Under the assumptions of proposition 8:*

$$\begin{aligned} & H^u\left(t_0, T_f, \{1, \ldots, m\}\right) \\ = & \sum_{j=1}^{m}\frac{AI_j\left(T_f\right)}{cf_j}\sum_{k=1}^{r+1} I_{j,k}\left[\Phi\left(z_{k-1}^*\right) - \Phi\left(z_k^*\right)\right] + \sum_{j=1}^{m}\sum_{i=1}^{N^j}\frac{k_i^j}{cf_j}F\left(t_0, T_f, T_i^j\right)w_{j,i}, \end{aligned} \tag{33}$$

where z_k^*, $k = 0, \ldots, r+1$, are given by definition 9, $\Phi(\cdot)$ represents the cumulative density function of the univariate standard normal distribution, $I_{j,k}$ is the indicator function

$$I_{j,k} := \mathbf{1}_{\left\{\frac{CB_j\left(T_f; z\right)}{cf_j} \leq \frac{CB_l\left(T_f; z\right)}{cf_l}, \forall l \neq j, \forall z \in \left[z_{k-1}^*, z_k^*\right]\right\}}, \tag{34}$$

and

$$w_{j,i} := \sum_{k=1}^{r+1} I_{j,k}\left\{\Phi\left[z_k^* - \tilde{\varphi}\left(t_0, T_f, T_i^j\right)\right] - \Phi\left[z_{k-1}^* - \tilde{\varphi}\left(t_0, T_f, T_i^j\right)\right]\right\}. \tag{35}$$

[11] Involving a single integration over the domain of Z.

Proof. Following Ritchken and Sankarasubramanian (1992, page 212), equation (26) can be rewritten as:

$$\begin{aligned} H^u(t_0, T_f, \{1, \ldots, m\}) &= \sum_{j=1}^{m} \int_{z \in S_j} dz \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right) \left\{ -\frac{AI_j(T_f)}{cf_j} \right. \\ &\quad \left. + \sum_{i=1}^{N^j} \frac{k_i^j}{cf_j} P\left(t_0, T_f, T_i^j\right) \exp\left[-\frac{1}{2}\tilde{\eta}\left(t_0, T_f, T_i^j\right) + \tilde{\varphi}\left(t_0, T_f, T_i^j\right) z\right] \right\}, \end{aligned}$$

where

$$S_j := \left\{ z \in \Re : \frac{CB_j(T_f; z)}{cf_j} \leq \frac{CB_l(T_f; z)}{cf_l}, \forall l \neq j \right\} \tag{36}$$

can be interpreted as the region of the state space for which the j^{th} deliverable bond is the time-T_f cheapest-to-deliver. Since the roots z_k^*, $k = 0, \ldots, r+1$, are assumed to be arranged in increasing order, and using definition (34), then

$$\begin{aligned} H^u(t_0, T_f, \{1, \ldots, m\}) &= \sum_{j=1}^{m} \sum_{k=1}^{r+1} I_{j,k} \int_{z_{k-1}^*}^{z_k^*} dz \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right) \left\{ -\frac{AI_j(T_f)}{cf_j} \right. \\ &\quad \left. + \sum_{i=1}^{N^j} \frac{k_i^j}{cf_j} P\left(t_0, T_f, T_i^j\right) \exp\left[-\frac{1}{2}\tilde{\eta}\left(t_0, T_f, T_i^j\right) + \tilde{\varphi}\left(t_0, T_f, T_i^j\right) z\right] \right\}. \end{aligned}$$

Expressing all integrands in terms of the univariate normal density function,

$$\begin{aligned} H^u(t_0, T_f, \{1, \ldots, m\}) &= -\sum_{j=1}^{m} \frac{AI_j(T_f)}{cf_j} \sum_{k=1}^{r+1} I_{j,k} \int_{z_{k-1}^*}^{z_k^*} dz \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right) \\ &\quad + \sum_{j=1}^{m} \sum_{i=1}^{N^j} \frac{k_i^j}{cf_j} P\left(t_0, T_f, T_i^j\right) \exp\left[-\frac{1}{2}\tilde{\eta}\left(t_0, T_f, T_i^j\right) + \frac{1}{2}\tilde{\varphi}\left(t_0, T_f, T_i^j\right)^2\right] \\ &\quad \sum_{k=1}^{r+1} I_{j,k} \int_{z_{k-1}^*}^{z_k^*} dz \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left[z - \tilde{\varphi}\left(t_0, T_f, T_i^j\right)\right]^2\right\}, \end{aligned}$$

and using proposition 6 as well as definitions (27) and (28), equation (33) arises. ■

Remark 9 *As in Carr and Chen (1997, equation 15), $w_{j,i}$ can be understood as weights such that $\sum_{j=1}^{m} w_{j,i} = 1, \forall i$, since definition (36) implies that $\bigcup_{j=1}^{m} S_j = \Re$. Therefore, the analytical pricing solution (33) can also be interpreted as (essentially) a weighted average of pure discount bond futures prices.*

5.1.2 Conditioning variable

The pricing formula (33) will only become a completely explicit solution after the specification of the conditioning random variable Z, which will define the deterministic function $\tilde{\varphi}\left(t_0, T_f, T_i^j\right)$. As argued, for instance, by Nielsen and Sandmann (2002, page 360), it is not possible to find the conditioning variable Z which minimizes the approximation error $H^u(t_0, T_f, \{1, \ldots, m\}) - H(t_0, T_f, \{1, \ldots, m\})$, that is with a perfect correlation with $\min_{j=1}^{m}\left[\frac{CB_j(T_f)}{cf_j}\right]$. Hence, Z will be chosen in order to simply enhance such correlation. Nevertheless, notice that even under the worst scenario of zero correlation, the upper bound $H^u(t_0, T_f, \{1, \ldots, m\})$ would be identical to the futures price without quality option, $H(t_0, T_f, \{j^*\})$.

Next proposition defines Z as being perfectly correlated[12] with $\frac{1}{m}\sum_{j=1}^{m}\frac{CB_j(T_f)}{cf_j}$. Therefore, the approximate pricing solution proposed by theorem 10 can be understood as arising from conditioning the minimum (of all deliverable bonds' time-T_f clean prices, corrected by its conversion factors) on the corresponding (arithmetic) average, in the same way as, for instance, Curran (1994) has conditioned the arithmetic average on the corresponding geometric mean.[13] Its accuracy will be (successfully) tested, in subsection 5.4, through a Monte Carlo experiment.

Proposition 11 *Under the assumptions of proposition 8, if*

$$Z = \frac{1}{\alpha}\sum_{u=1}^{m}\sum_{v=1}^{N^u}\frac{k_v^u}{cf_u}P(t_0, T_f, T_v^u)\int_{t_0}^{T_f}\left[\underline{\sigma}(s, T_v^u) - \underline{\sigma}(s, T_f)\right]' \cdot d\underline{W}^{\mathbb{Q}}(s), \tag{37}$$

where

$$\alpha^2 := \sum_{u=1}^{m}\sum_{v=1}^{N^u}\sum_{p=1}^{m}\sum_{q=1}^{N^p}\frac{k_v^u k_q^p}{cf_u cf_p}P(t_0, T_f, T_v^u)\,P\left(t_0, T_f, T_q^p\right)\psi\left(t_0, T_f, T_v^u, T_q^p\right), \tag{38}$$

with

$$\psi\left(t_0, T_f, T_v^u, T_q^p\right) := \int_{t_0}^{T_f}\left[\underline{\sigma}(s, T_v^u) - \underline{\sigma}(s, T_f)\right]' \cdot \left[\underline{\sigma}\left(s, T_q^p\right) - \underline{\sigma}(s, T_f)\right] ds, \tag{39}$$

then

$$Z \backsim N^1(0, 1), \tag{40}$$

$$\tilde{\varphi}\left(t_0, T_f, T_i^j\right) = \frac{1}{\alpha}\sum_{u=1}^{m}\sum_{v=1}^{N^u}\frac{k_v^u}{cf_u}P(t_0, T_f, T_v^u)\,\psi\left(t_0, T_f, T_v^u, T_i^j\right), \tag{41}$$

and[14]

$$corr\left[\frac{1}{m}\sum_{j=1}^{m}\frac{CB_j(T_f)}{cf_j}, Z\right] \approx 1. \tag{42}$$

Proof. From equation (38), since $\alpha^2 = \mathbb{E}_{\mathbb{Q}}\left[(\alpha Z)^2\middle|\mathcal{F}_{t_0}\right]$, with Z defined through equation (37), then condition (40) is verified.

Using definition (27),

$$\begin{aligned}\tilde{\varphi}\left(t_0, T_f, T_i^j\right) &= \frac{1}{\alpha}\sum_{u=1}^{m}\sum_{v=1}^{N^u}\frac{k_v^u}{cf_u}P(t_0, T_f, T_v^u)\,\mathbb{E}_{\mathbb{Q}}\left\{\int_{t_0}^{T_f}\left[\underline{\sigma}(s, T_v^u) - \underline{\sigma}(s, T_f)\right]' \cdot d\underline{W}^{\mathbb{Q}}(s)\right.\\ &\left.\int_{t_0}^{T_f}\left[\underline{\sigma}\left(s, T_i^j\right) - \underline{\sigma}(s, T_f)\right]' \cdot d\underline{W}^{\mathbb{Q}}(s)\middle|\mathcal{F}_{t_0}\right\},\end{aligned}$$

which yields equation (41) after considering definition (39).

[12] Up to a first-order approximation.

[13] Intuitively, the conversion factors' scheme should ensure a positive correlation between $\min_{j=1}^{m}\left[\frac{CB_j(T_f)}{cf_j}\right]$ and $\frac{1}{m}\sum_{j=1}^{m}\frac{CB_j(T_f)}{cf_j}$, by reducing the dispersion of the "m" underlying random variables.

[14] $corr(X, Y)$ represents the linear correlation coefficient between the random variables X and Y.

Finally, proposition 5 and definitions (16) and (17) imply that

$$
\begin{aligned}
& \frac{1}{m}\sum_{j=1}^{m}\frac{CB_j\left(T_f\right)}{cf_j}-\mathbb{E}_{\mathcal{Q}}\left[\left.\frac{1}{m}\sum_{j=1}^{m}\frac{CB_j\left(T_f\right)}{cf_j}\right|\mathcal{F}_{t_0}\right] \qquad (43)\\
= & \frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{N^j}\frac{k_i^j}{cf_j}P\left(t_0,T_f,T_i^j\right)\left\{\exp\left[-\frac{1}{2}\eta\left(t_0,T_f,T_i^j\right)+\int_{t_0}^{T_f}\left(\underline{\sigma}\left(s,T_i^j\right)-\underline{\sigma}\left(s,T_f\right)\right)'\cdot d\underline{W}^{\mathcal{Q}}\left(s\right)\right]\right.\\
& \left.-\exp\left[-\frac{1}{2}\eta\left(t_0,T_f,T_i^j\right)+\frac{1}{2}\varphi\left(t_0,T_f,T_i^j\right)\right]\right\}.
\end{aligned}
$$

Applying the first-order approximation $\exp(x)\approx 1+x$ to equation (43), while using equation (38), then[15]

$$
\begin{aligned}
& cov\left[\frac{1}{m}\sum_{j=1}^{m}\frac{CB_j\left(T_f\right)}{cf_j},Z\right]\\
\approx & \mathbb{E}_{\mathcal{Q}}\left\{\left[\frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{N^j}\frac{k_i^jP\left(t_0,T_f,T_i^j\right)\left(-\frac{\varphi\left(t_0,T_f,T_i^j\right)}{2}+\int_{t_0}^{T_f}\left(\underline{\sigma}\left(s,T_i^j\right)-\underline{\sigma}\left(s,T_f\right)\right)'\cdot d\underline{W}^{\mathcal{Q}}\left(s\right)\right)}{cf_j}\right]\right.\\
& \left.\left.\left[\frac{1}{\alpha}\sum_{u=1}^{m}\sum_{v=1}^{N^u}\frac{k_v^uP\left(t_0,T_f,T_v^u\right)}{cf_u}\int_{t_0}^{T_f}\left(\underline{\sigma}\left(s,T_v^u\right)-\underline{\sigma}\left(s,T_f\right)\right)'\cdot d\underline{W}^{\mathcal{Q}}\left(s\right)\right]\right|\mathcal{F}_{t_0}\right\}\\
= & \frac{\alpha}{m},
\end{aligned}
$$

and

$$
\mathbb{E}_{\mathcal{Q}}\left\{\left.\left[\frac{1}{m}\sum_{j=1}^{m}\frac{CB_j\left(T_f\right)}{cf_j}-\mathbb{E}_{\mathcal{Q}}\left(\left.\frac{1}{m}\sum_{j=1}^{m}\frac{CB_j\left(T_f\right)}{cf_j}\right|\mathcal{F}_{t_0}\right)\right]^2\right|\mathcal{F}_{t_0}\right\}\approx\frac{\alpha^2}{m^2},
$$

which justifies result (42), since Z possesses a unit variance. ■

5.2 Proportionality assumption

The second approximation proposed for futures prices with an embedded quality option is based on the following result.

Proposition 12 *Under the HJM model (9), the time-t price of a pure discount bond with maturity T $(\geq t)$ is equal in distribution, under the equivalent martingale measure $\mathcal{Q}$, to:*[16]

$$
P\left(t,T\right)\overset{d}{=}P\left(t_0,t,T\right)\exp\left[-\frac{1}{2}\eta\left(t_0,t,T\right)+\sqrt{\varphi\left(t_0,t,T\right)}z\right], \qquad (44)
$$

with $z\backsim N^1\left(0,1\right)$.

[15] $cov\left(X,Y\right)$ represents the covariance between the random variables X and Y.

[16] The symbol $\overset{d}{=}$ represents equality in distribution.

Proof. Replacing T_f by t in equation (11),

$$\begin{aligned} P(t,T) &= P(t_0,t,T)\exp\left\{-\frac{1}{2}\int_{t_0}^{t}\left[\left\|\underline{\sigma}(s,T)\right\|^2-\left\|\underline{\sigma}(s,t)\right\|^2\right]ds \right. \\ &\quad \left. +\int_{t_0}^{t}\left[\underline{\sigma}(s,T)-\underline{\sigma}(s,t)\right]'\cdot d\underline{W}^{Q}(s)\right\}. \end{aligned} \tag{45}$$

Using definition (16) and the probability law (20), equation (44) arises. ■

An approximate analytical pricing solution can be obtained for Treasury bond futures by assuming that equation (44) is valid not only as an equality in distribution but also as an equality in value. Such approximation is in the spirit of the proportionality assumption used by El Karoui and Rochet (1989, page 22) or of the *rank 1 approximation* suggested by Brace and Musiela (1994, equation 6.1), which have both been shown to be accurate in the context of European swaption pricing -see, for instance, Brace and Musiela (1994, table 7.5). Although based on completely different assumptions, next corollary shows that such approximate pricing formula possesses the same structure as the one obtained in theorem 10 through the conditioning approach.

Corollary 13 *Assuming that equation (44) is valid not only as an equality in distribution but also as an equality in value, the time-t_0 fair price, $H(t_0,T_f,\{1,\ldots,m\})$, of a bond futures contract that matures at time T_f ($\geq t_0$) and is written on a delivery basket containing m deliverable Treasury coupon-bearing bonds can be approximated through theorem 10, but with $\tilde{\varphi}\left(t_0,T_f,T_i^j\right)$ replaced by $\sqrt{\varphi\left(t_0,T_f,T_i^j\right)}$, for $j=1,\ldots,m$ and $i=1,\ldots,N^j$.*

Proof. Combining equations (1) and (22),

$$H(t_0,T_f,\{1,\ldots,m\})=\mathbb{E}_{Q}\left\{\min_{j=1}^{m}\left[-\frac{AI_j(T_f)}{cf_j}+\sum_{i=1}^{N^j}\frac{k_i^j}{cf_j}P\left(T_f,T_i^j\right)\right]\middle|\mathcal{F}_{t_0}\right\},$$

and approximating equation (44) as an equality in value,

$$\begin{aligned} & H(t_0,T_f,\{1,\ldots,m\}) \\ \approx\ & \mathbb{E}_{Q}\left\{\min_{j=1}^{m}\left[-\frac{AI_j(T_f)}{cf_j}+\sum_{i=1}^{N^j}\frac{k_i^j P\left(t_0,T_f,T_i^j\right)\exp\left(-\frac{\eta\left(t_0,T_f,T_i^j\right)}{2}+\sqrt{\varphi\left(t_0,T_f,T_i^j\right)}z\right)}{cf_j}\right]\middle|\mathcal{F}_{t_0}\right\}. \end{aligned} \tag{46}$$

Comparing equations (26) and (46), corollary 13 follows. ■

From a theoretical point of view, there are, at least, two reasons for the *proportionality assumption* to be less appealing than the *conditioning approach*: the approximation error of the former methodology should increase with n; and, the sign of the approximation error is unknown. Concerning the relation between both methodologies, corollary 13 implies that the proposed approximations would be equivalent if and only if

$$\tilde{\varphi}\left(t_0,T_f,T_i^j\right)=\sqrt{\varphi\left(t_0,T_f,T_i^j\right)},\forall i,j. \tag{47}$$

Combining equations (38) and (41),

$$\tilde{\varphi}\left(t_0, T_f, T_i^j\right)^2 = \frac{\sum_{u=1}^{m}\sum_{v=1}^{N^u}\sum_{p=1}^{m}\sum_{q=1}^{N^p}\frac{k_v^u k_q^p P\left(t_0,T_f,T_v^u\right)P\left(t_0,T_f,T_q^p\right)}{cf_u cf_p}\psi\left(t_0,T_f,T_v^u,T_i^j\right)\psi\left(t_0,T_f,T_q^p,T_i^j\right)}{\sum_{u=1}^{m}\sum_{v=1}^{N^u}\sum_{p=1}^{m}\sum_{q=1}^{N^p}\frac{k_v^u k_q^p P\left(t_0,T_f,T_v^u\right)P\left(t_0,T_f,T_q^p\right)}{cf_u cf_p}\psi\left(t_0,T_f,T_v^u,T_q^p\right)},$$

it follows that, if[17]

$$\psi\left(t_0, T_f, T_v^u, T_i^j\right)\psi\left(t_0, T_f, T_q^p, T_i^j\right) \approx \psi\left(t_0, T_f, T_v^u, T_q^p\right)\varphi\left(t_0, T_f, T_i^j\right), \forall i, j, u, v, p, q, \quad (48)$$

then the equivalence condition (47) will be obtained. However, equation (48) is only exact under a single-factor specification ($n = 1$). Nevertheless, the Monte Carlo study presented in subsection 5.4 shows that both approximations are, for different parameter constellations and for different contract specifications, almost indistinguishable.

5.3 Nested time-homogeneous specification

Although theorem 10 and corollary 13 are valid for any deterministic volatility specification (even for time-inhomogeneous or non-Markovian models), the subsequent empirical analysis will be cast into a simpler time-independent framework. In fact, since the goal is only to fit market prices of Treasury bond futures, such time-homogeneous setup should be sufficient to recover the main principal diagonal elements of the market interest rate covariance matrix. Of course, and as Rebonato (1998, page 70) argues, it will be extremely difficult to fit the market interest rate correlation structure through a low dimensional and time-independent HJM Gauss-Markov model. Nevertheless, our purpose is not to price or hedge interest rate correlation dependent derivatives and, therefore, the following time-homogeneity restriction should not be too severe.

The Gauss-Markov time-homogeneous HJM model that will be estimated is defined by equation (9) and through the following proposition.

Proposition 14 *If the short-term interest rate is Markovian and the volatility function $\underline{\sigma}(\cdot, T) : [t_0, T] \to \Re^n$ is time-homogeneous, then the volatility function must be restricted to the following analytical specification:*

$$\underline{\sigma}(t, T)' := \underline{G}' \cdot a^{-1} \cdot \left[I_n - e^{a(T-t)}\right], \quad (49)$$

where $I_n \in \Re^{n \times n}$ represents an identity matrix, while $\underline{G} \in \Re^n$ and $a \in \Re^{n \times n}$ contain the model' time-independent parameters.

Proof. Proposition 14 follows, for instance, from Musiela and Rutkowski (1998, proposition 13.3.2). ■

Using the volatility specification (49), appendix A provides explicit solutions for several time-integrals contained in the previously derived pricing formulae.

[17] Notice that $\psi\left(t_0, T_f, T_i^j, T_i^j\right) = \varphi\left(t_0, T_f, T_i^j\right)$.

5.4 Monte Carlo study

In order to test the accuracy of the approximate analytical pricing solutions proposed in theorem 10 and corollary 13, a Monte Carlo experiment will be run. Approximate futures prices will be compared against exact futures prices, which are obtained through Monte Carlo simulation with antithetic variates.

In order to compute Monte Carlo exact price estimates, forward zero-coupon bond prices -as given by equation (11)- are subject to a Euler discretization, under measure $\mathcal{Q}$. At each simulation, the discretized version of equation (11) is evolved, from the valuation date (time t_0) and until the expiry date of the futures contract (time T_f), for all the maturities T ($\geq T_f$) that correspond to cash flow payment dates of all the underlying deliverable bonds. For that purpose, and on each time-step, a set of n independent normal and antithetic variates is generated through the Box-Muller algorithm. Since $P(T_f, T_f, T) = P(T_f, T)$, at each simulation and for the last time-step, the futures contract (with an embedded quality option) can be valued -at time T_f- through equation (3). Finally, and according to equation (1), the Monte Carlo estimate for the exact time-t_0 price of the futures contract is simply obtained by computing the arithmetic average of the time-T_f futures' values generated at all simulations. For all the futures contracts to be tested bellow, Monte Carlo simulations are run with 520 time steps per year and until an accuracy (ratio between the standard error and the price estimate) of one basis point is achieved.

The inputs needed to run the Monte Carlo study (initial yield curve, volatility parameters, deliverable bonds, etc.) correspond to three different days selected randomly from our data set[18]: 31/Aug/99, 25/Feb/00 and 10/May/00. For each valuation date, a different futures contract was selected: a long-term one on 31/Aug/99 (Bloomberg code RXZ9), a medium-term one on 25/Feb/00 (Bloomberg code OEU0) and a short-term contract on 10/May/00 (Bloomberg code DUZ0). The delivery basket underlying each futures contract is described in Table 1. Because the quality option value is small for all the selected dates (as well as for all the data set, as will be shown in subsection 7.3) and in order to test the accuracy of the proposed approximations under different market conditions, a fictitious futures contract (labeled "ADUZ0") was also created through the augmentation of the delivery basket underlying the short-term contract DUZ0 (last column of Table 1). Moreover, such artificial contract will be valued not only on 10/May/00 but also one year earlier (10/May/99) in order to further enhance the corresponding quality option value.

For each date, the term structure of default-free spot interest rates was computed through the methodology that will be described in section 6, which will also be shown to provide *consistent* estimates for the volatility model' parameters contained in (a diagonal) matrix a. Figure 1 summarizes the spot yield curves obtained for each day, which are all upward slopped. Then, and again for each cross-section, the HJM model' additional parameters that also parameterize the volatility functions (vector $\underline{G}$) are estimated by minimizing the mean absolute percentage differences between the model -as given by theorem 10 and proposition 11- and the market prices of all traded bond futures contracts. The estimated parameters, for a three-dimensional model' specification, are contained in Table 2.

Table 3 presents the results of the Monte Carlo study. For each date, futures prices (with embedded quality options) are approximated through theorem 10 and proposition 11 ("upper bound" price) and via corollary 13 ("*rank 1*" price) in less than one second.[19] For all the contracts, the difference between the two approximations under analysis (column "*rank 1* diff.") is completely negligible.

[18] To be described in subsections 6.2 and 7.1.

[19] Throughout this paper, all computations were made by running Pascal programs on an Intel Xeon 2.80 GHz processor.

Exact futures prices (with embedded quality options) are obtained through antithetic Monte Carlo simulations, which are much more time consuming than the proposed approximations. Standard errors (bellow 0.01%) of the Monte Carlo price estimates are also provided. Pricing errors are defined as the differences between approximate ("upper bound"column) and exact futures prices with embedded quality options. Finally and in order to measure the importance of the quality option feature on each date, each futures price is also computed assuming the existence of no delivery options (via proposition 7). The quality option is then expressed as the difference between exact futures prices without and with quality option features, divided by the Monte Carlo price estimate.

For all the dates considered, the pricing errors obtained are very small (less than one basis point of the exact price) and well inside the Monte Carlo standard errors. Such impressive accuracy is observed not only for the traded contracts (RXZ9, OEU0, and DUZ0), where the quality option value is small (and no larger than 0.83%), but also for the artificial contract "ADUZ0". Hence, the accuracy of the proposed approximations is not confined to the EUREX futures contracts, which are characterized by a narrow delivery basket and a small time-to-maturity. On 10/May/00, the "ADUZ0" fictitious futures contract (with 17 deliverable issues) presents a much higher quality option value: 3.45%. Moreover, the same contract but valued on 10/May/99 (although based on the model' parameters and yield curve data prevailing on 10/May/00), that is with one additional year of time-to-maturity, yields an even larger estimate for the quality option: 6.69%. Nevertheless, for all dates and for all contracts, the accuracy (and efficiency) of the two proposed approximations is excellent.

In summary, the approximate analytical pricing solution proposed in theorem 10 and proposition 11 seems to be extremely accurate and fast to implement (as well as almost indistinguishable from the *rank 1* approximation of corollary 13) and will, consequently, be used in the forthcoming empirical analysis.

6 Consistent forward rate curves

6.1 Restrictions on the volatility functions

In order to price interest rate contingent claims under the term structure model (9) it is necessary to obtain two model' inputs: the initial forward rate curve and the parameters' values defining the volatility function (49).

Concerning the first input, many different parametric functional forms can be used in order to estimate discount factors from the observed market prices of Treasury (coupon-bearing) bonds -see, for instance, McCulloch (1971) and Nelson and Siegel (1987), or Jeffrey, Linton and Nguyen (2000) for a survey. However, and as argued by Bjork and Christensen (1999), the choice of the functional form describing the initially "observed" forward interest rate curve should depend on the formulation adopted -equation (9)- for the term structure model under use (in terms of both the number of Brownian motions and the volatility specification considered). That is, the family $\mathcal{G}$ of forward rate curves used for model' recalibration (e.g. exponential splines) must be *consistent* with the dynamics implied by the interest rate model $\mathcal{M}$ under use, in the sense that, given an initially observed forward rate curve in $\mathcal{G}$, the interest rate model $\mathcal{M}$ should only produce forward rate curves belonging to the same manifold $\mathcal{G}$.

Bjork and Christensen (1999, page 327) point out two reasons for such consistency requirement to be empirically relevant. Firstly, if a given interest rate model $\mathcal{M}$ is supposed to be subject to daily recalibration[20], it is important that, on each day, the parametrized family of forward rate curves $\mathcal{G}$,

[20] As it will be the case for the empirical analysis of the quality option to be presented in section 7.

that is fitted to bond market data, is general enough to be invariant under the dynamics of the term structure model; otherwise, the marking to market of an interest rate derivative would yield value changes due not to interest rate movements but rather to model' inconsistencies. Secondly, if a specific family $\mathcal{G}$ of forward rate curves is shown to be able to efficiently recover the cross-section of bond prices observed in the market, then it makes sense to incorporate such implied yield behavior in the dynamics of the interest rate model $\mathcal{M}$ under use. In practical terms, and as proposition 15 will reveal, the consistency between $\mathcal{M}$ and $\mathcal{G}$ will be ensured by simply incorporating in the volatility function (49) some parameters (matrix a) that are estimated through the best fitting of the initially observed forward rate curve.

Proposition 15 defines the functional form of the forward rate curve used to fit the initially observed term structure of interest rates, which is shown to be the most parsimonious linear-exponential parameterization that is consistent with the term structure model (9).

Proposition 15 *Under the assumption that matrix a is diagonal, the minimal consistent family (manifold) $\mathcal{G}$ of forward rate curves which is invariant under the dynamics of the Gaussian and time-homogeneous HJM model specified by equations (9) and (49) is defined through the mapping $\gamma : \Re^{2n} \times \Re_{+} \rightarrow \Re$ such that*

$$\gamma\left(\underline{z}, x\right) := f\left(t, t + x\right) = \sum_{j=1}^{n} z_j \exp\left(a_j x\right) + \sum_{j=1}^{n} z_{n+j} \exp\left(2 a_j x\right), \tag{50}$$

where z_j represents the j^{th} element of vector $\underline{z} \in \Re^{2n}$, a_j defines the j^{th} principal diagonal element of matrix a, and $f\left(t, t+x\right)$ corresponds to the time-t instantaneous forward interest rate for date $(t + x)$, with $x \in \Re_{+}$.

Proof. See appendix B. ■

Remark 10 *Concerning matrix a, and as argued by Duan and Simonato (1995, page 26), the "assumption of diagonability does not involve an appreciable loss of generality" because the eigenvalues of a matrix are continuous functions of its elements (and thus, multiple roots of the characteristic equation can be avoided by a small adjustment in the original matrix). Moreover, parameter's restrictions on the off-principal diagonal elements of matrix a will prove useful in order to ensure the stability of the estimates obtained through the fitting of bond market prices.*

Equipped with the consistent specification of the forward rate curve provided by proposition 15, parameters a and $\underline{z}$ can be estimated by minimizing the sum of absolute percentage differences between a cross-section of market Treasury coupon-bearing bond prices and the corresponding discounted values obtained by decomposing each government bond into a portfolio of pure discount bonds, which are parameterized as

$$P\left(t, T\right) = \exp\left\{\sum_{j=1}^{n} \frac{z_j}{a_j}\left[1 - e^{a_j (T-t)}\right] + \sum_{j=1}^{n} \frac{z_{n+j}}{2 a_j}\left[1 - e^{2 a_j (T-t)}\right]\right\}. \tag{51}$$

Equation (51) follows immediately from proposition 15 by integrating definition (57), that is, using the fact that $P\left(t, T\right) = \exp\left[-\int_t^T f\left(t, u\right) du\right]$.

Remark 11 *In this section, starting from a postulated dynamic for the term structure of interest rates -equations (9) and (49)- a consistent parameterization was found for the discount function*

-equation (51). De Rossi (2002), also based on Bjork and Christensen (1999, theorem 4.1), has independently implemented the inverse procedure. Starting from an augmented Nelson and Siegel (1987) specification for the forward rate curve, he has shown that the consistent dynamics of the yield curve must be represented by the extended Vasicek (1977) model described in Hull and White (1990). Moreover, De Rossi (2002) has also proved that a restricted exponential specification for the forward rate curve implies the interest rate dynamics offered by the generalized Vasicek (1977) model described in Hull and White (1994). Filipović (1999) had already shown that no stochastic term structure model is consistent with the original Nelson and Siegel (1987) specification.

6.2 Bonds' data set: description and empirical results

The data used to estimate the spot yield curve was obtained from Bloomberg. It consists of (bid and ask) prices, recorded at the end of each exchange session, for the 116 German Treasury (coupon-bearing) bonds that were traded on the EUREX and on the Frankfurt Stock Exchange, over the period between 4/May/99 and 28/Sep/01.

In order to minimize the well known problems associated with distorted prices from bonds that were not actively traded during the sample period, an ad-hoc filter has been used to exclude from the sample all the issues that possess a relatively large bid-ask spread[21] compared to the bid-ask spread that was typical found for bonds with similar residual maturity.[22] After imposing the above defined liquidity filter, the average sample size for a cross-section in years 1999, 2000 and 2001, become equal to 65, 48 and 57 bonds, respectively.

On each sample day, the estimation of the term structure of interest rates was made by finding the values of the parameters $\underline{z} \in \Re^{2n}$ and $\{a_1, \ldots, a_n\}$ -contained in equation (51)- that minimize the absolute percentage differences between fitted and market coupon-bearing bond prices. Consistently with the analysis that will be presented in subsection 6.3, the HJM model' dimension will be set to three factors ($n = 3$) and, therefore, nine parameters were used in the discount function (51). More specifically, on any day t, parameters $\underline{z} \in \Re^6$ and $\{a_1, a_2, a_3\}$ were estimated through the minimization of the following mean absolute percentage error measure for the quality of fit:

$$MAPE\left(t\right) = \frac{\sum_{j=1}^{m(t)} \left| \frac{CB_j(t) - \left[\sum_{i=1}^{N^j} P\left(t, T_i^j\right) k_i^j - AI_j(t)\right]}{CB_j(t)} \right|}{m\left(t\right)}, \tag{52}$$

where $CB_j\left(t\right)$ is the observed average bid-ask (clean) price for the j^{th} bond in the sample, N^j is the number of cash flows k_i^j ($i = 1, \ldots, N^j$) paid by that bond at times T_i^j ($> t$), $AI_j\left(t\right)$ represents the interest accrued by the same bond at time t, and $m\left(t\right)$ is the number of Treasury bonds considered in the time-t cross-sectional optimization. The objective function' dependence on the parameters is obtained through the discount factors $P\left(t, T_i^j\right)$, which are computed from equation (51). Throughout this paper all optimization routines are based on the quasi-Newton method, with backtracking line searches, described in Dennis and Schnabel (1996, section 6.3).

[21] Defined as the difference between ask and bid clean prices, divided by their average.

[22] Three homogeneous time-to-maturity classes have been considered: a first one with a time-to-maturity below 7.5 years and presenting an average bid-ask spread of 0.05%; a second class with a time-to-maturity between 7.5 and 15 years, with an average bid-ask spread of 0.065%; and, a third class, that groups maturities above 15 years, which presents an average bid-ask spread of 0.095%. Whenever the bid-ask spread of a specific bond (on a given day) was observed to be greater than the average spread of its class, such bond has been automatically excluded from the cross-section under analysis.

The results obtained show that the estimation methodology adopted fits rather well the discount function implicit in the German government bond market, resulting in reliable and smooth yield curves for the sample period under observation. In order to validate the previous assertion, the minimized MAPE values are presented in Figure 2, for all the cross-sections under analysis. The maximum in-sample MAPE value (0.1635%) was observed in 29/Jun/00 and the minimum (0.0481%) occurred in 24/Jan/00. The sample average MAPE is only equal to 0.0877%, which is only about 2 basis points above the average in-sample bid-ask spread. Figure 3 compares market and fitted bond prices for the cross-section corresponding to the maximum MAPE observed, and shows small absolute pricing errors.

Figure 4 shows the estimated spot yield surface for maturities between one and 15 years, with only two observations per month (on Wednesdays), during the sample period under analysis (from 4/May/99 to 28/Sep/01). Table 4 presents several descriptive statistics of the spot interest rates estimated for some representative maturities and during the sample period under analysis, which includes 626 daily observations. During almost all the sample period, the spot yield curve presented a positive slope, although it was approximately flat between August and November 2000. Nevertheless, it is clear that the period under analysis contains a wide variety of term structure shapes. As usual, the shorter maturity rates present a higher volatility than the longer maturity ones.

Finally, Table 5 summarizes the sample correlation matrix of interest rates daily changes, which will be used (in the next subsection) to define the dimension of the HJM term structure model under analysis. As expected, the movements amongst all interest rates are not perfectly correlated: the linear correlation coefficients are higher for contiguous maturities and much lower between the extreme points of the yield curve.

6.3 Principal components analysis

In the previous subsection, the spot yield surface was fitted through a parametric function that is consistent with the interest rate dynamics generated by a three-factor HJM model. The goal of the present subsection is to provide empirical support that confirms the choice of three as the minimum number of non-trivial factors needed to reproduce almost all of the interest rates variance structure. For that purpose, a principal components analysis (PCA) will be implemented.

The data consists of daily estimated continuously compounded spot interest rates for sixteen maturities, between six months and 15 years, yielding a total of 626×16 data points, from 4/May/99 to 28/Sep/01. The PCA was performed not on interest rate levels but rather on daily interest rate changes, since the latter were checked to be stationary. Table 6 presents the first seven eigenvalues (in a strictly decreasing order) and the corresponding (orthogonal) eigenvectors of the sample correlation matrix. The first column shows the maturities of the spot rates that were considered. The remaining columns in the table show the correspondent factors (or "principal components") describing the interest rate movements.

The first factor (column ΔZ_1) is related to parallel shifts in the yield curve. This factor is made up by approximately equal weights (with the same sign) and can be intuitively interpreted as an average *level* component. The second factor (column ΔZ_2) is made up by weights of similar magnitude and opposite signs at the opposite ends of the maturity spectrum. It corresponds to a "twist" or "steepening" of the yield curve and can be interpreted as a *slope* factor. The third factor (column ΔZ_3) corresponds to a "bowing" of the yield curve because it is made of weights of similar magnitude and identical signs at the extremes of the maturity spectrum, contrasting with the middle maturity loadings. This feature permits the interpretation of the third component as the *curvature* factor.

The second, third, and fourth rows of Table 6 show the eigenvalues and the corresponding contribution of the different principal components for the explanation of the overall interest rate variability. The first factor explains 75.9% of the total sample variance. The second and third factors possess a much lower explanatory power: 13.8% and 6.6%, respectively. Consequently, the first three factors, taken together, span more than 96% of the interest rate variability. Therefore, the number of independent linear combinations needed to summarize the dynamics of the yield curve, in its entirety, can be reduced, without much loss of information, to only three orthogonal factors. Since the previous empirical analysis suggests that a three-factor term structure model could describe well enough the sample variation of the yield curve, the valuation of the quality option will be, hereafter, cast into a three-factor HJM Gaussian framework.

7 Empirical analysis of the quality option implicit in the EUREX market

7.1 Futures' data set description

This section estimates the implicit quality option value embedded in the EUREX Treasury bond futures contracts, during the sample period between May 1999 and September 2001, and using the HJM framework presented in sections 4 and 5. Futures prices were gently provided by the EUREX Statistical Department, while the deliverable sets were captured from Bloomberg.

There are two main reasons for focusing the empirical analysis on the EUREX Treasury bond futures contracts: the extreme liquidity of these contracts, on one hand, and the fact that there are no other embedded delivery options to consider but the quality option.

Concerning the first argument, the EUREX Euro-Bund Future is one of the world's most heavily traded futures contract. Accordingly to EUREX' statistics[23], in 2001, the Euro-Bund Future accounted for more than 178 million contracts traded. During the same period, the CBOT traded approximately 58 million US T-Bond futures contracts and Euronext traded only 7 million Euro Notional Futures contracts. Moreover, the Schatz (short-term) and the Bobl (medium-term) futures are the world's most heavily traded contracts in the 2 and 5 years segments, respectively. EUREX' high volumes are generated from trading on a full range of interest rate products that cover the whole German yield curve (from one month to 30 years). The reduction of interest rate differentials between debt instruments from the various European Union members implied the reduction of basis risk, which has lead to the consolidation of the EUREX Euro-Bund Future as the benchmark derivative contract for all euro-denominated government debt issues.

The second reason is that, unlike the CBOT T-bond futures, the EUREX Euro-Bund Future possesses no timing options. Therefore, the valuation framework presented in section 2 can be exactly applied without neglecting or separating the value of the timing options.

In order to estimate the magnitude of the quality option embedded in the German Treasury bond futures contracts, we have collected the daily average bid-ask prices and the specifications of 39 futures contracts traded at EUREX, from 4/May/99 through 28/Sep/01. During 614 days, a total of 5,439 market prices were gathered. Tables 7, 8 and 9 summarize the main characteristics of those contracts. Each day contains (at most) nine active contracts, corresponding to three contract' sets with different delivery cycles: three short-term, three medium-term and three long-term contracts, all of them maturing in a quarterly cycle within March, June, September and December.

[23] Figures were obtained from EUREX (2002).

7.2 Estimation methodology

For each sample day, and based on the term structure of interest rates previously estimated in subsection 6.2, the HJM volatility parameters $\underline{G}$ -see equation (49)- are inferred by minimizing the mean absolute percentage differences between model' futures values -as given by theorem 10 and proposition 11- and market prices. As shown in Figure 5, the time-homogeneous volatility function (49) ensures a remarkable fit between model' futures values (with embedded quality options) and observed market prices: the average daily absolute percentage pricing error was only about 8 basis points (ranging from 0.015% to 0.424%). Then, on each day, the estimated spot yield curve and the volatility function (implicit in the market for Treasury bond futures contracts) are used to compute futures values without (that is, not deducted from) quality option features, through equation (21). On each day, and for each futures contract, the quality option value is given by the percentage difference between equations (33) and (21). Before applying the previously described quality option' estimation methodology, some data-mining problems were dealt with, namely: liquidity deficiencies of the futures contracts (measured through traded volumes); and, the treatment of the so called *new issue option*.

In general, liquidity is observed to be almost null from the issue date of the futures contract to 30 weeks prior to the delivery month. After this period the contract' liquidity increases substantially and rapidly until its maturity. Whenever there are no trades for a futures contract, the representativeness of its price must be questioned because it is computed by EUREX through the cost-of-carry model (that is as if it were a forward contract). In fact, and as noticed by El Karoui et al. (1991), the forward price overestimates the true Treasury bond futures value and, consequently, it should not be used to price the quality option[24]. To obviate this problem, we have recovered the daily volumes of all futures contracts and avoided those prices that were not produced by market trades, that is, we have excluded all the zero volume prices (828 observations). Table 10 presents the descriptive statistics that were obtained for the quality option estimates, both with zero volume prices excluded and also with the all sample considered. Although the average value of the quality option is very low in both series, it is clear that the series including the zero volume prices over-estimates the value of the quality option.

The second problem concerns the existence of -what Lin and Paxson (1995) call- the *new issue quality option*, which arises whenever the exchange allows new bond issues to be included in the futures contracts' delivery basket after the first trading day. This is the case for the Treasury bond futures contracts traded on EUREX, and therefore the final list of deliverable bonds (and corresponding conversion factors) can, in the limit, be only known on the last trading day. Consequently, on each day (and for a specific futures contract) the quality option value is derived not only from the possibility of choice amongst the currently known delivery set but it should also be affected (to a much less extent) by the probability of new bond issues being integrated into the delivery basket. Because it is not possible to know in advance which new bond issues will be incorporated, by the exchange, in the deliverable set, we are left with two alternatives. One possibility would be to value, on each trading day, a futures contract based on the deliverable set that will prevail on the last trading day. Such valuation alternative was not pursued because it would distort the quality option value, since it incorrectly assumes that the market is able to exactly guess, in advance, a delivery basket that will only be revealed on the last trading day. The second alternative, that was implemented, consists in valuing each futures contract, on each trading day, only based on the currently known deliverable set. That is, a new bond issue is only considered in the valuation of a futures contract when it is integrated, by the exchange, in the deliverable basket. In other words, the *new issue* component of the total *quality option* value has been ignored in the present empirical

[24] Chen (1997, proposition 5) also argued that futures prices are bounded from above by the cost-of-carry model.

analysis.

In summary, zero volume futures prices have been excluded from the data sample, and the deliverable set varies on a daily basis for the same futures contract.

7.3 Empirical results

The estimates of the quality option values are based on the (non-zero volume) $4,611$ futures prices captured for the 614 days covering the period from 4/May/99 to 28/Sep/01. Table 10 shows that, when compared with previous empirical studies about the US Treasury bond futures market, the EUREX' quality option is quite irrelevant: its global average value is only about 0.048% of the futures price (ranging from zero to a maximum of 0.83%), which is well inside the daily average futures pricing absolute percentage error (see Figure 5). Moreover, our estimates yielded a highly significative number of zero quality option values (the median is equal to 1.12×10^{-09}), which reinforces the assertion of an insignificant impact of the quality option contractual feature on the market futures prices.

One possible explanation for the insignificance of the quality option estimates is the existence of a small number of (homogeneous) deliverable issues underlying the EUREX Treasury bond futures contracts: between a minimum of two and a maximum of eight, our sample contains an average of five coupon-bearing deliverable bonds per contract -see Tables 7, 8 and 9. For instance, Hemler (1990) describes more than thirty US Treasury deliverable bonds (varying widely in terms of coupon rates and time-to-maturity) for the CBOT T-bond futures contracts. Another plausible explanation for the low significance of the quality option is the possibility that futures' buyers tend to attenuate the bid-down pressure on futures prices because they can eliminate the "quality risk" by off-setting their positions (just) before expiration. Figure 6 shows the open interest evolution one week prior to the expiry date, for eight delivery cycles and for short-, medium- and long-term futures contracts. It is clear that the largest part of the open interest is off-set during the last week. Furthermore, one day prior to expiration, on average, only about 15% of the maximum open interest is still alive and possible awaiting for physical delivery. Finally, it can also be argued that the EUREX' quality option is of European style (because there is only one pre-specified delivery day for each contract) whereas the CBOT' quality option is of Bermudan style (since the futures' seller can choose the delivery day amongst any trading day of the delivery month) and, therefore, more valuable.

Table 11 decomposes the quality option estimates by contract type. The average futures prices and average monetary values of the quality option for the whole sample period are given at the bottom of the same table. The long-term contract presents the highest quality option value: 83 basis points (b.p.) or EUR 877.4 in terms of the average futures price. The medium-term contract possesses the highest estimated average quality option value (6.4 b.p., corresponding to EUR 66.7), which can be explained by two reasons. First, at expiration, the average number of Treasury bonds that are available for delivery in the medium-term contract (6) is higher than the average number for the long-term contract (4.5) -see Tables 8 and 9. Second, in the medium-term contract the final delivery basket is more predictable than in the long-term contract and, thus, market operators may incorporate part of the new issue option value in the futures prices. The short-term contract presents the lowest estimated average quality option value: only about 1.79 basis points or EUR 18.3. This feature is in accordance with Ritchken and Sankarasubramanian (1995), although it can no longer be explained by the existence of a significantly larger number of deliverable bonds in the long-term contracts' deliverable sets. The lower volatility of short-term futures prices could be a possible explanation for the lower quality option value observed in those short-term contracts.

Option theory predicts that the quality option value should be higher for a longer time-to-maturity of the futures contract, because the uncertainty regarding the choice of the cheapest-to-deliver bond on the delivery date should, also, be higher. To test this "negative theta feature" hypothesis, we have computed the weekly average percentage values of the quality option, which are plotted in Figure 7. The first data-range in the "weeks prior to delivery" axis includes the average quality option (percentage) value prior to the 20^{th} week before the delivery date, the second data-range presents the average estimate computed between the 15^{th} and the 20^{th} week prior to the delivery date and so on. From the 15^{th} week until the delivery date, the quality option estimates decline steadily for all the three types of contracts analyzed. Moreover, fifteen weeks prior to maturity, the quality option value is quite small and ranges from 3 basis points in the short-term contract to approximately 8 basis points in the long-term contract. The global average quality option value equals only 6 basis points. These results are consistent with the 9 b.p. estimate obtained by Lin and Paxson (1995, page 114), three months before the delivery date and for the LIFFE market.[25] Four weeks prior to delivery, the quality option value practically disappears for the short-term contract and declines substantially in the medium and long-term contracts. The long-term futures contract is the one whose estimated quality option value is more persistent: between the 15^{th} and the 4^{th} week prior to the delivery date, the quality option value only declines 35.6%. For the medium-term and the short-term futures contracts, the quality option value declines 65.7% and 80.8%, respectively, during the same period.

8 Conclusions

The main theoretical contribution of the present work consisted in deriving, for the first time to the authors' knowledge, an analytical (but approximate) pricing solution for the quality option embedded in Treasury bond futures contracts, under a multi-factor Gaussian HJM framework. Using a *conditioning* or a *rank 1* approximation, and no matter the diversity of the underlying delivery basket or the dimension of HJM model under analysis, the fair value of a Treasury bond futures contract (with an embedded quality option) was written as a weighted average of pure discount bond futures prices, where the weights simply involved the standard univariate normal distribution function.

In order to test the accuracy of the approximate analytical pricing solutions proposed in theorem 10 and corollary 13, a Monte Carlo experiment was run. For different parameters's constellations and for different contract specifications, the pricing errors obtained were very small (less than one basis point of the exact price) and well inside the Monte Carlo standard errors, as well as almost indistinguishable between both approximations. Therefore, the fast and accurate pricing solution proposed in theorem 10 and proposition 11 was then used in the subsequent empirical analysis.

Concerning the dimensionality of the Gauss-Markov HJM model, a principal components analysis was applied and the usual three stylized factors were found (*level*, *slope* and *curvature*). Before testing the empirical significance of the quality option in the EUREX derivatives market, the term structure of default-free interest rates was estimated, on each day of the sample period, through a *consistent* parametrization of the discount function, in the sense of Bjork and Christensen (1999). Proposition 15 establishes the family of forward rate curves which is invariant under the dynamics of the HJM model under analysis. Such specification was then fitted to the market prices of German Treasury coupon-bearing bonds, producing low percentage pricing errors (of about, on average, 8.77 basis points).

[25] The estimates obtained by Lin and Paxson (1995) are from a time period (1987-1991) of relatively high interest rate volatility when, *ceteris paribus*, option values should be higher. Our sample covers a period (1999-2001) of much lower interest rate volatility.

Finally, the significance of the quality option was tested through the calibration of a three-factor and time-homogeneous HJM specification to the Treasury bond futures contracts traded at EUREX during the period between 4/May/99 and 28/Sep/01, and two empirical findings were highlighted: the excellent fit obtained to the market prices of Treasury bond futures contracts (with an average absolute percentage daily error of only 8 basis points); and, the small average value estimated for the quality option. The empirical evidence suggests that the magnitude of the quality option is quite irrelevant for the EUREX' German Treasury bond futures market (and specially for short-term contracts), which contradicts the majority of the empirical studies previously devoted to the US T-bond futures contracts. In our data set, the most expressive estimate of the quality option is only equal to 83 b.p. of the futures price, while the mean percentage value is even lower than 5 basis points.

A Appendix: Gauss-Markov time-homogeneous specification

Next proposition recasts definitions (16), (17) and (39) under a Markovian and time-independent HJM framework.

Proposition 16 *Under the volatility specification (49),*

$$\begin{aligned} \eta\left(t_0,t,T\right) &= 2\underline{G}'\cdot a^{-1}\cdot\left(a^{-1}\right)'\cdot\left[\underline{\sigma}\left(t_0,T\right)-\underline{\sigma}\left(t,T\right)-\underline{\sigma}\left(t_0,t\right)\right] \\ &\quad +\underline{G}'\cdot a^{-1}\cdot\left[\Delta\left(t_0,T\right)-\Delta\left(t,T\right)-\Delta\left(t_0,t\right)\right]\cdot\left(a^{-1}\right)'\cdot\underline{G}, \end{aligned} \tag{53}$$

$$\psi\left(t_0,T_f,T_v^u,T_q^p\right)=\underline{\sigma}\left(T_f,T_v^u\right)'\cdot\Delta\left(t_0,T_f\right)\cdot\underline{\sigma}\left(T_f,T_q^p\right), \tag{54}$$

and

$$\varphi\left(t_0,t,T\right)=\underline{\sigma}\left(t,T\right)'\cdot\Delta\left(t_0,t\right)\cdot\underline{\sigma}\left(t,T\right), \tag{55}$$

where

$$\Delta\left(t,T\right):=\left(a+a'\right)^{-1}\cdot\left[e^{(a+a')(T-t)}-I_n\right]. \tag{56}$$

Proof. Available upon request. ■

Remark 12 *All the matrix exponentials involved in the previous formulae can be computed using Padé approximations with scaling and squaring. For details, see Van Loan (1978).*

B Appendix: Proof of proposition 15

Equation (50) arises as a straightforward application of the two locally invariance conditions imposed by Bjork and Christensen (1999, theorem 4.1). In order to take advantage of such elegant result, the term structure model under analysis will be first rewritten in terms of instantaneous forward interest rates. Using equation (9) and applying Itô's lemma to the following definition

$$f\left(t,T\right):=-\frac{\partial\ln P\left(t,T\right)}{\partial T}, \tag{57}$$

the HJM model under consideration can be equivalently specified as

$$df\left(t,T\right)=\frac{\partial\underline{\sigma}\left(t,T\right)'}{\partial T}\cdot\underline{\sigma}\left(t,T\right)dt-\frac{\partial\underline{\sigma}\left(t,T\right)'}{\partial T}\cdot d\underline{W}^{\mathbb{Q}}\left(t\right). \tag{58}$$

Furthermore, the time-homogeneous specification (49) adopted for the volatility function $\underline{\sigma}(t,T)$ and the diagonability assumption imply that

$$df(t,T) = \underline{G}' \cdot e^{a(T-t)} \cdot \left[e^{a'(T-t)} - I_n\right] \cdot \left(a^{-1}\right)' \cdot \underline{G}dt + \sum_{j=1}^{n} G_j \exp\left[a_j (T-t)\right] dW_j^{\mathcal{Q}}(t), \qquad (59)$$

where G_j denotes the j^{th} element of the parameter vector $\underline{G}$, and $W_j^{\mathcal{Q}}(t)$ is the j^{th} Brownian motion contained in $\underline{W}^{\mathcal{Q}}(t)$.

From Bjork and Christensen (1999, equation 4.11), one of the invariance conditions to be met by the HJM specification (59) is that

$$\underline{G}' \cdot e^{ax} \in Im\left[\gamma_{\underline{z}}(\underline{z}, x)\right], \qquad (60)$$

i.e. the vector of volatilities for $df(t, t+x)$ must be contained in the image of the Fréchet derivative of $\gamma(\underline{z}, x)$ with respect to $\underline{z}$. Using definition (50), condition (60) is verified if and only if there exist some constants $\alpha_i \in \Re$ $(i = 1, \ldots, 2n)$ such that

$$G_j \exp(a_j x) = \sum_{i=1}^{n} \alpha_i \exp(a_i x) + \sum_{i=1}^{n} \alpha_{n+i} \exp(2a_i x), \qquad (61)$$

for $j = 1, \ldots, n$. This is clearly the case as long as $\alpha_j = G_j$ and $\alpha_i = 0$ for $i \neq j$.

Concerning the second consistency condition, as given by Bjork and Christensen (1999, equation 4.10), it is necessary that

$$\left[\gamma_x(\underline{z}, x) + \underline{G}' \cdot e^{ax} \cdot \int_0^x e^{a's} ds \cdot \underline{G}\right] \in Im\left[\gamma_{\underline{z}}(\underline{z}, x)\right], \qquad (62)$$

where γ_x represents the Fréchet derivative of $\gamma(\underline{z}, x)$ with respect to x. Or, since matrix a is assumed to be diagonal, condition (62) can be restated as

$$\begin{aligned} & \sum_{j=1}^{n} a_j z_j \exp(a_j x) + \sum_{j=1}^{n} 2a_j z_{n+j} \exp(2a_j x) + \sum_{j=1}^{n} \frac{G_j^2}{a_j}\left[\exp(2a_j x) - \exp(a_j x)\right] \\ = \; & \sum_{i=1}^{n} \beta_i \exp(a_i x) + \sum_{i=1}^{n} \beta_{n+i} \exp(2a_i x), \end{aligned} \qquad (63)$$

for some constants $\beta_i \in \Re$ $(i = 1, \ldots, 2n)$. The above equality is easily verified if $\beta_j = a_j z_j - \frac{G_j^2}{a_j}$ for $j \leq n$, and if $\beta_j = 2a_{j-n} z_j + \frac{G_{j-n}^2}{a_{j-n}}$ for $n < j \leq 2n$. ■

References

Arnold, L., 1992, *Stochastic Differential Equations: Theory and Applications* (Krieger Publishing Company, Florida).

Barnhill, T., 1990, Quality Option Profits, Switching Option Profits and Variation Margin Costs: An Evaluation of their Size and Impact on Treasury Bond Futures Prices, *Journal of Financial and Quantitative Analysis* 25, 65–86.

Barnhill, T. and W. Seale, 1988, Optimal Exercises of the Switching Option in Treasury Bond Arbitrages, *Journal of Futures Markets* 8, 517–532.

Benninga, S. and M. Smirlock, 1985, An Empirical Analysis of the Delivery Option, Marking to Market and the Pricing of the Treasury Bond Futures, *Journal of Futures Markets* 5, 361–74.

Bick, A., 1997, Two Closed Formulas for the Futures Price in the Presence of a Quality Option, *European Finance Review* 1, 81–104.

Bjork, T. and B. Christensen, 1999, Interest Rate Dynamics and Consistent Forward Rate Curves, *Mathematical Finance* 9, 323–349.

Boyle, P., 1989, The Quality Option and Timing Option in Futures Contracts, *Journal of Finance* 44, 101–113.

Brace, A. and M. Musiela, 1994, A Multifactor Gauss Markov Implementation of Heath, Jarrow, and Morton, *Mathematical Finance* 4, 259–283.

Bühler, W., K. Düllmann, and M. Windfuhr, 2001, Affine Models, Credit Spreads and the Delivery Option of a Multi-Issuer Bond Future, Working paper, University of Mannheim.

Carr, P. and R.-R. Chen, 1997, Valuing Bond Futures and the Quality Option, Working paper, Morgan Stanley and Rutgers University.

Chance, D. and M. Hemler, 1993, The Impact of Delivery Options on Futures Prices: A Survey, *Journal of Futures Markets* 13, 127–155.

Chen, R.-R., 1997, Bounds for Treasury Bond Futures Prices and Embedded Delivery Options: Theory and Empirical Analysis, Working paper, Rutgers University.

Cox, J., J. Ingersoll, and S. Ross, 1981, The Relation Between Forward Prices and Futures Prices, *Journal of Financial Economics* 9, 321–346.

Cox, J., J. Ingersoll, and S. Ross, 1985a, An Intertemporal General Equilibrium Model of Asset Prices, *Econometrica* 53, 363–384.

Cox, J., J. Ingersoll, and S. Ross, 1985b, A Theory of the Term Structure of Interest Rates, *Econometrica* 53, 385–407.

Curran, M., 1994, Valuing Asian and Portfolio Options by Conditioning on the Geometric Mean Price, *Management Science* 40, 1705–1711.

De Rossi, G., 2002, Kalman Filtering of Consistent Forward Rate Curves: A Tool to Estimate and Model Dynamically the Term Structure, Working paper, University of Cambridge.

Dennis, J. and R. Schnabel, 1996, *Numerical Methods for Unconstrained Optimization and Nonlinear Equations* (SIAM, Philadelphia).

Duan, J.-C. and J.-G. Simonato, 1995, Estimating and Testing Exponential-Affine Term Structure Models by Kalman Filter, Working paper, McGill University.

Duffie, D., 1989, *Futures Markets* (Prentice-Hall, Englewood Cliffs, New Jersey).

El Karoui, N. and J.-C. Rochet, 1989, A Pricing Formula for Options on Coupon Bonds, Working paper 72, SEEDS.

El Karoui, N., C. Lepage, R. Myneni, N. Roseau, and R. Viswanathan, 1991, The Valuation and Hedging of Contingent Claims with Markovian Interest Rates, Working paper, Université de Paris VI.

EUREX, Eurex - The European Derivatives Market, Eurex-Communications 2002.

Filipović, D., 1999, A Note on the Nelson-Siegel Family, *Mathematical Finance* 9, 349–359.

Gay, G. and S. Manaster, 1984, The Quality Option Implicit in Futures Contracts, *Journal of Financial Economics* 13, 353–370.

Geman, H., N. El Karoui, and J.-C. Rochet, 1995, Changes of Numeraire, Changes of Probability Measure and Option Pricing, *Journal of Applied Probability* 32, 443–458.

Heath, D., R. Jarrow, and A. Morton, 1990, Bond Pricing and the Term Structure of Interest Rates: A Discrete Time Approximation, *Journal of Financial and Quantitative Analysis* 25, 419–440.

Heath, D., R. Jarrow, and A. Morton, 1992, Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation, *Econometrica* 60, 77–105.

Hegde, S., 1988, An Empirical Analysis of Implicit Delivery Options in T-Bond Futures Contracts, *Journal of Banking and Finance* 12, 469–92.

Hemler, M., 1990, The Quality Delivery Option in Treasury Bond Futures Contracts, *Journal of Finance* 45, 1565–1586.

Hull, J. and A. White, 1990, Pricing Interest Rate Derivative Securities, *Review of Financial Studies* 3, 573–592.

Hull, J. and A. White, 1994, Numerical Procedures for Implementing Term Structure Models II: Two-Factor Models, *Journal of Derivatives*, Winter, 37–48.

Jeffrey, A., O. Linton, and T. Nguyen, 2000, Flexible Term Structure Estimation: Which Method Is Preferred?, Working paper, Yale International Center for Finance.

Kane, A. and A. Marcus, 1986, The Quality Option in the Treasury Bond Futures Market: An Empirical Assessment, *Journal of Futures Markets* 6, 231–248.

Karlin, S. and H. Taylor, 1975, *A First Course in Stochastic Processes*, second ed. (Academic Press, New York).

Kennedy, D., 1994, The Term Structure of Interest Rates as a Gaussian Random Field, *Mathematical Finance* 4, 247–258.

Lamberton, D. and B. Lapeyre, 1996, *Introduction to Stochastic Calculus Applied to Finance* (Chapman & Hall/CRC, London).

Lin, B. and D. Paxson, 1995, Term Structured Volatility and Bund Futures Embedded Options, *Journal of Business Finance and Accounting* 22, 101–126.

Litterman, R. and J. Scheinkman, 1991, Common Factors Affecting Bond Returns, *Journal of Fixed Income* 1, 54–61.

Livingston, M., 1987, The Delivery Option on Forward Contracts, *Journal of Financial and Quantitative Analysis* 22, 79–87.

Longstaff, F. and E. Schwartz, 1992, Interest Rate Volatility and the Term Structure: A Two-Factor General Equilibrium Model, *Journal of Finance* 47, 1259–1282.

Margrabe, W., 1978, The Value of an Option to Exchange One Asset for Another, *Journal of Finance* 33, 177–186.

McCulloch, J., 1971, Measuring the Term Structure of Interest Rates, *Journal of Business* 44, 19–31.

Mood, A., F. Graybill, and D. Boes, 1974, *Introduction to the Theory of Statistics*, third ed. (McGraw-Hill, Singapore).

Musiela, M. and M. Rutkowski, 1998, *Martingale Methods in Financial Modelling*, Applications of Mathematics: Stochastic Modelling and Applied Probability 36 (Springer-Verlag, Berlin Heidelberg).

Nelson, C. and A. Siegel, 1987, Parsimonious Modeling of Yield Curves, *Journal of Business* 60, 473–489.

Nielsen, J. and K. Sandmann, 2002, Pricing of Asian Exchange Rate Options under Stochastic Interest Rates as a Sum of Options, *Finance and Stochastics* 6, 355–370.

Nielsen, J. and K. Sandmann, 2003, Pricing Bounds on Asian Options, *Journal of Financial and Quantitative Analysis* 38, 449–473.

Nunes, J., 2004, Multi-Factor Valuation of Floating Range Notes, *Mathematical Finance* 14, 79–97.

Nunes, J. and L. Clewlow, 1999, Kalman Filtering of Gaussian Exponential-Affine LIBOR-Rate Models Using Caps and Swaptions, Working paper, Warwick Business School, University of Warwick.

Pang, K., 1998, Calibration of Gaussian Heath, Jarrow and Morton and Random Field Interest Rate Term Structure Models, *Review of Derivatives Research* 2, 315–345.

Press, W., B. Flannery, S. Teukolsky, and W. Vetterling, 1994, *Numerical Recipes in Pascal: The Art of Scientific Computing* (Cambrigde University Press, Cambridge).

Rebonato, R., 1998, *Interest-Rate Option Models: Understanding, Analysing and Using Models for Exotic Interest-Rate Options*, second ed. (John Wiley & Sons, Chichester).

Revuz, D. and M. Yor, 1991, *Continuous Martingales and Brownian Motion* (Springer-Verlag, Berlin Heidelberg).

Ritchken, P. and L. Sankarasubramanian, 1992, Pricing the Quality Option in Treasury Bond Futures, *Mathematical Finance* 2, 197–214.

Ritchken, P. and L. Sankarasubramanian, 1995, A Multifactor Model of the Quality Option in Treasury Futures Contracts, *Journal of Financial Research* 18, 261–279.

Rogers, L. and Z. Shi, 1995, The Value of an Asian Option, *Journal of Applied Probability* 32, 1077–1088.

Van Loan, C., 1978, Computing Integrals Involving the Matrix Exponential, *IEEE Transactions on Automatic Control* 23, 395–404.

Vasicek, O., 1977, An Equilibrium Characterization of the Term Structure of Interest, *Journal of Financial Economics* 5, 177–188.

Table 1: Monte Carlo study - deliverable bonds underlying the futures contracts under analysis

Bond issue	Coupon	Maturity	Conversion factors			
(j)	rate	date	RXZ9	OEU0	DUZ0	ADUZ0
EC085455	3.750%	04/Jan/09	0.846008			*0.859315*
EC114151	4.000%	04/Jul/09	0.856929			*0.870102*
EC155787	4.500%	04/Jul/09	0.892856			*0.901481*
EC003630	4.750%	04/Jul/08	0.917801			*0.925419*
EC060679	4.125%	04/Jul/08	0.876915			*0.888343*
GG726558	6.000%	05/Jan/06		0.999620		*0.999888*
GG726572	6.000%	16/Feb/06		0.999571		*0.999737*
GG726097	6.500%	14/Oct/05		1.021232		*1.020231*
GG725379	6.875%	12/May/05		1.034271		*1.032601*
EC228806	5.000%	20/May/05		0.959348		*0.962088*
GG714280	7.250%	21/Oct/02			1.021150	1.021150
GG714720	7.125%	20/Dec/02			1.020814	1.020814
GG729514	4.500%	18/Feb/03			0.969867	0.969867
GG729363	5.000%	12/Nov/02			0.982249	0.982249
GG714144	7.750%	01/Oct/02			1.028774	1.028774
GG714856	7.125%	29/Jan/03			1.021680	1.021680
GG714576	7.375%	02/Dec/02			1.024865	1.024865
Nr. of deliverable bonds (m)			5	5	7	17
Futures' delivery date (T_f)			10/Dec/99	11/Sep/00	11/Dec/00	11/Dec/00

Contracts and bonds are identified by their Bloomberg' codes. Contract ADUZ0 is fictitious and corresponds to the enlargement of the delivery basket associated to the traded contract DUZ0. Conversion factors in italic are not published by the EUREX but rather computed as the unit face value clean prices of the deliverable bonds, on the delivery date, such that the yields-to-maturity of all deliverable issues are equal to 6%.

Table 2: Monte Carlo study - three-factor Gauss-Markov HJM volatility functions for the dates under analysis

Valuation	Vector $\underline{G}$			Diagonal matrix a		
date	G_1	G_2	G_3	a_{11}	a_{22}	a_{33}
31/Aug/99	-3.612E-07	0.0528709	1.949E-07	-2.8525515	-0.0066896	-2.5571848
25/Feb/00	-2.889E-05	0.0362698	2.703E-05	-4.2243907	-0.0245879	-3.5073576
10/May/00	1.158E-05	-0.0263357	1.952E-05	-4.2741229	-0.0083193	-3.3833823

Vector $\underline{G} \in \Re^3$ and matrix $a \in \Re^{3\times3}$ are defined through proposition 14. All parameters were estimated from the German Treasury bonds and futures prices, as described in sections 6 and 7.

Table 3: Monte Carlo study - results

Valuation date	Futures contracts	Analytical solutions			Antithetic Monte Carlo			
		Exact price without QO	Approx. price with QO Upper bound	Approx. price with QO *Rank 1* diff.	Exact price with QO	Quality option	Standard errors	Pricing errors
31/Aug/99	RXZ9	107.30	106.43	0.00E+00	106.42	0.83%	0.010	0.004
			(0.16s)	(0.11s)	(2,374.07s)			
25/Feb/00	OEU0	102.64	102.07	2.33E-10	102.07	0.56%	0.010	0.003
			(0.10s)	(0.08s)	(224.31s)			
10/May/00	DUZ0	101.56	101.34	1.33E-10	101.33	0.22%	0.008	0.007
			(0.06s)	(0.06s)	(10.52s)			
10/May/00	ADUZ0	101.56	98.18	-2.53E-10	98.18	3.45%	0.010	-0.003
			(0.33s)	(0.19s)	(16,512.22s)			
10/May/99	ADUZ0	100.77	94.46	-1.56E-10	94.46	6.69%	0.009	0.004
			(0.67s)	(0.53s)	(86,997.98s)			

Contracts are identified by their Bloomberg' codes. Contract ADUZ0 is fictitious and corresponds to the enlargement of the delivery basket associated to the traded contract DUZ0. The last line values the artificial contract ADUZ0 but one year earlier, although based on the model' parameters and the spot yield curve prevailing on 10/May/00. Exact futures price without quality option (QO) is given by proposition 7. Approximate analytical futures prices with quality option (QO) are given by theorem 10, for the upper bound, and by corollary 13, for the rank 1 approximation. Column "rank 1 diff." presents the difference between the above two approximated futures prices. Monte Carlo simulations are run with 520 time steps per year and until an accuracy (ratio between the standard error and the price estimate) of one basis point is achieved. The quality option is expressed as the percentage difference between exact futures prices without and with QO. Pricing errors are the differences between the "upper bound" analytical approximation and exact Monte Carlo futures prices. The CPU times are reported in parenthesis and expressed in seconds.

Table 4: Summary statistics for some continuously compounded spot interest rates

	6 Months	1 Year	5 Years	10 Years	15 Years
Mean	4.012%	4.098%	4.635%	5.026%	5.296%
Median	4.282%	4.200%	4.617%	5.086%	5.323%
St. Deviation	0.759%	0.730%	0.422%	0.292%	0.220%
Minimum					
value	2.548%	2.538%	3.298%	4.030%	4.560%
date	13/May/99	13/May/99	04/May/99	04/May/99	04/May/99
Maximum					
value	5.090%	5.359%	5.279%	5.595%	5.814%
date	31/Oct/00	21/Aug/00	19/May/00	18/Jan/00	04/Jan/00

Sample period: 4/May/99 - 28/Sep/01.

Table 5: Linear correlation coefficients amongst daily changes of continuously compounded spot interest rates

	0.5y	1y	2y	3y	4y	5y	7y	10y	15y
0.5y	1.0000								
1y	0.6774	1.0000							
2y	0.5144	0.8420	1.0000						
3y	0.4710	0.7369	0.9672	1.0000					
4y	0.4298	0.6883	0.9299	0.9895	1.0000				
5y	0.3915	0.6607	0.8985	0.9673	0.9927	1.0000			
7y	0.3267	0.6222	0.8383	0.9081	0.9508	0.9797	1.0000		
10y	0.2604	0.5608	0.7349	0.8022	0.8488	0.8866	0.9458	1.0000	
15y	0.1579	0.2976	0.3834	0.4461	0.4569	0.4567	0.4979	0.7189	1.0000

Sample period: 4/May/99 - 28/Sep/01.

Table 6: Principal Component Analysis of continuously compounded spot rate daily changes

	ΔZ_1	ΔZ_2	ΔZ_3	ΔZ_4	ΔZ_5	ΔZ_6	ΔZ_7
Eigenvalues	12.142	2.209	1.059	0.320	0.240	0.029	0.000
Explained Variance	75.9%	13.8%	6.6%	2.0%	1.5%	0.2%	0.0%
Cumulative	75.9%	89.7%	96.3%	98.3%	99.8%	100.0%	100.0%
Original Variables				Eigenvectors			
0.5y	-0.110	0.325	0.683	-0.597	0.217	-0.109	-0.012
1y	-0.194	0.314	0.409	0.370	-0.649	0.366	0.051
2y	-0.245	0.289	0.073	0.454	0.092	-0.694	-0.361
3y	-0.260	0.235	-0.033	0.262	0.372	-0.021	0.583
4y	-0.267	0.202	-0.110	0.099	0.330	0.293	0.196
5y	-0.271	0.176	-0.168	-0.026	0.196	0.312	-0.184
6y	-0.273	0.143	-0.202	-0.117	0.051	0.209	-0.314
7y	-0.275	0.099	-0.212	-0.174	-0.073	0.075	-0.265
8y	-0.278	0.044	-0.198	-0.201	-0.161	-0.046	-0.125
9y	-0.279	-0.024	-0.160	-0.199	-0.209	-0.133	0.039
10y	-0.279	-0.102	-0.102	-0.169	-0.213	-0.176	0.174
11y	-0.274	-0.184	-0.026	-0.115	-0.172	-0.169	0.242
12y	-0.263	-0.264	0.060	-0.042	-0.092	-0.115	0.218
13y	-0.245	-0.333	0.146	0.040	0.013	-0.026	0.100
14y	-0.224	-0.386	0.224	0.121	0.130	0.083	-0.094
15y	-0.201	-0.423	0.288	0.193	0.243	0.197	-0.337

Sample period: 4/May/99 - 28/Sep/01.

Table 7: List of short-term futures contracts included in the sample database

Contract (tick)	Delivery date	Number of deliverable bonds	Average maturity (years)	Average coupon rate
DUH0*	10/Mar/00	4	1.94	5.25%
DUH1*	12/Mar/01	8	2.00	6.02%
DUH2	11/Mar/02	4	2.07	4.88%
DUM0*	12/Jun/00	4	1.99	5.38%
DUM1*	11/Jun/01	8	1.98	5.72%
DUM2	10/Jun/02	5	2.05	5.88%
DUM9	10/Jun/99	4	2.07	6.78%
DUU0*	11/Sep/00	7	2.01	6.41%
DUU1*	10/Sep/01	8	1.94	5.44%
DUU9	10/Sep/99	5	1.97	5.95%
DUZ0*	11/Dec/00	8	1.96	6.39%
DUZ1	10/Dec/01	6	1.96	4.79%
DUZ9	10/Dec/99	5	1.96	5.80%

The sample contains 13 short-term Treasury bond futures contracts traded at EUREX over the period from 4/May/99 to 28/Sep/01. Average maturities and average coupon rates were computed by a simple arithmetic average of the maturities and coupons of all deliverable bonds underlying each contract. Contracts are identified by their Bloomberg' code. The contracts signaled with an asterisk have a complete cycle within the sample period.

Table 8: List of medium-term futures contracts included in the sample database

Contract (tick)	Delivery date	Number of deliverable bonds	Average maturity (years)	Average coupon rate
OEH0*	10/Mar/00	7	4.75	5.98%
OEH1*	12/Mar/01	5	4.88	5.95%
OEH2	11/Mar/02	3	5.02	5.33%
OEM0*	12/Jun/00	7	4.76	5.96%
OEM1*	11/Jun/01	4	4.70	5.81%
OEM2	10/Jun/02	3	4.77	5.33%
OEM9	10/Jun/99	6	5.02	6.33%
OEU0*	11/Sep/00	6	5.02	5.90%
OEU1*	10/Sep/01	3	4.96	5.58%
OEU9	10/Sep/99	7	4.92	6.61%
OEZ0*	11/Dec/00	5	5.03	5.95%
OEZ1	10/Dec/01	2	4.88	5.25%
OEZ9	10/Dec/99	8	4.73	5.94%

The sample contains 13 medium-term Treasury bond futures contracts traded at EUREX over the period from 4/May/99 to 28/Sep/01. Average maturities and average coupon rates were computed by a simple arithmetic average of the maturities and coupons of all deliverable bonds underlying each contract. Contracts are identified by their Bloomberg' code. The contracts signaled with an asterisk have a complete cycle within the sample period.

Table 9: List of long-term futures contracts included in the sample database

Contract (tick)	Delivery date	Number of deliverable bonds	Average maturity (years)	Average coupon rate
RXH0*	10/Mar/00	4	9.32	4.41%
RXH1*	12/Mar/01	3	9.32	5.29%
RXH2	11/Mar/02	3	9.32	5.08%
RXM0*	12/Jun/00	5	9.26	4.58%
RXM1*	11/Jun/01	4	9.32	5.22%
RXM2	10/Jun/02	3	9.07	5.08%
RXM9	10/Jun/99	5	9.27	4.38%
RXU0*	11/Sep/00	4	9.19	4.78%
RXU1*	10/Sep/01	3	9.32	5.17%
RXU9	10/Sep/99	6	9.48	4.42%
RXZ0*	11/Dec/00	5	9.16	4.88%
RXZ1	10/Dec/01	3	9.07	5.17%
RXZ9	10/Dec/99	6	9.23	4.42%

The sample contains 13 long-term Treasury bond futures contracts traded at EUREX over the period from 4/May/99 to 28/Sep/01. Average maturities and average coupon rates were computed by a simple arithmetic average of the maturities and coupons of all deliverable bonds underlying each contract. Contracts are identified by their Bloomberg' code. The contracts signaled with an asterisk have a complete cycle within the sample period.

Table 10: Quality option estimates - both for the all sample and also without considering zero volume observations (restricted sample)

	Futures exact price without QO		Futures approximated price with QO		Quality Option (QO)	
	Restricted	All sample	Restricted	All sample	Restricted	All sample
Mean	104.82%	104.66%	104.77%	104.60%	0.0479%	0.0579%
Median	104.00%	104.00%	104.00%	104.00%	1.12E-09	9.49E-07
St. Deviation	2.50%	2.46%	2.49%	2.47%	0.1046%	0.1050%
Minimum	101.00%	101.00%	101.00%	101.00%	-5.55E-12	-7.37E-12
Maximum	116.00%	116.00%	116.00%	116.00%	0.8250%	0.9480%
Nr. observations	4,611	5,439	4,611	5,439	4,611	5,439

On each day, the sample consists of 39 Treasury bond futures contracts, traded at EUREX over the period from 4/May/99 to 28/Sep/01. Futures exact analytical prices without QO are given by proposition 7. Futures approximate analytical prices with QO are given by theorem 10 and proposition 11. The quality option (QO) is expressed as the percentage difference of the above mentioned futures prices.

Table 11: Average embedded quality option (QO) value by contract type (zero volume observations excluded)

	Global	Short-Term	Medium-Term	Long-Term
Mean	0.0479%	0.0179%	0.0636%	0.0569%
Median	1.12E-09	-3.56E-12	2.50E-07	-4.08E-12
Standard Deviation	0.1046%	0.0379%	0.1194%	0.1193%
Maximum	0.8250%	0.2470%	0.7840%	0.8250%
Avg Futures Price	104.77%	102.64%	104.80%	106.36%
Standard Deviation	2.49%	1.10%	1.87%	2.53%
Avg QO Value (EUR)	50.2	18.3	66.7	60.5
Max QO Value (EUR)	864.3	253.5	821.6	877.4
Nr. observations	4,611	1,325	1,535	1,751

Sample period: 4/May/99 - 28/Sep/01.

Figure 1: Continuously compounded spot yield curves used for the Monte Carlo study

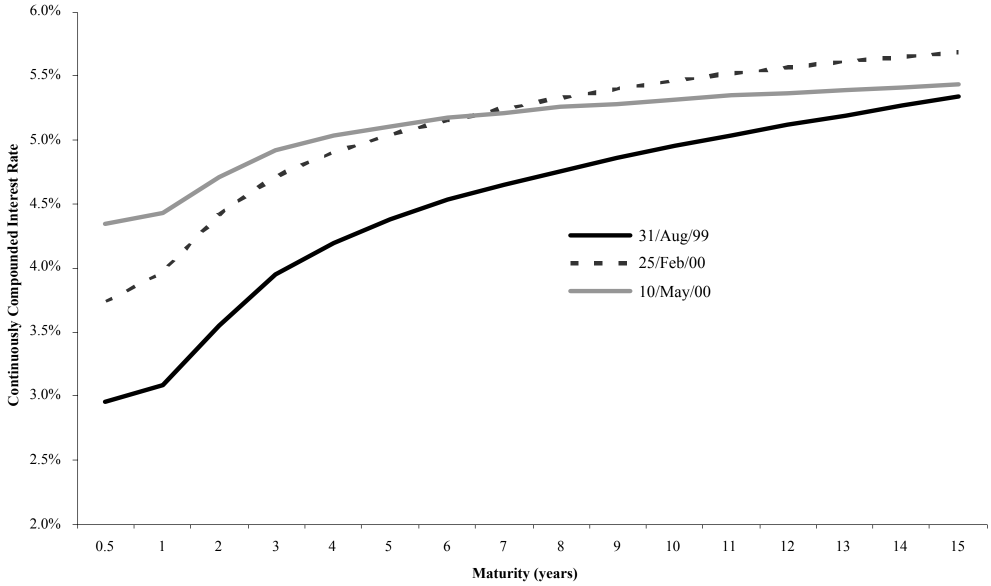

Figure 2: In-sample mean absolute percentage errors (MAPE) for the German government bond market - 4/May/99 to 28/Sep/01

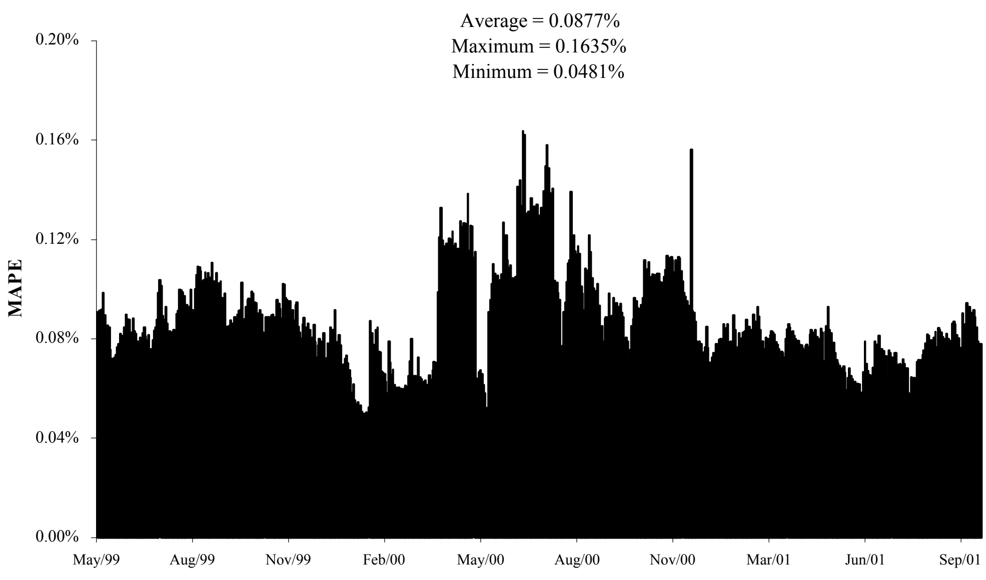

Figure 3: Market versus fitted bond prices for the cross-section (29/Jun/00) with the highest MAPE (0.1635%)

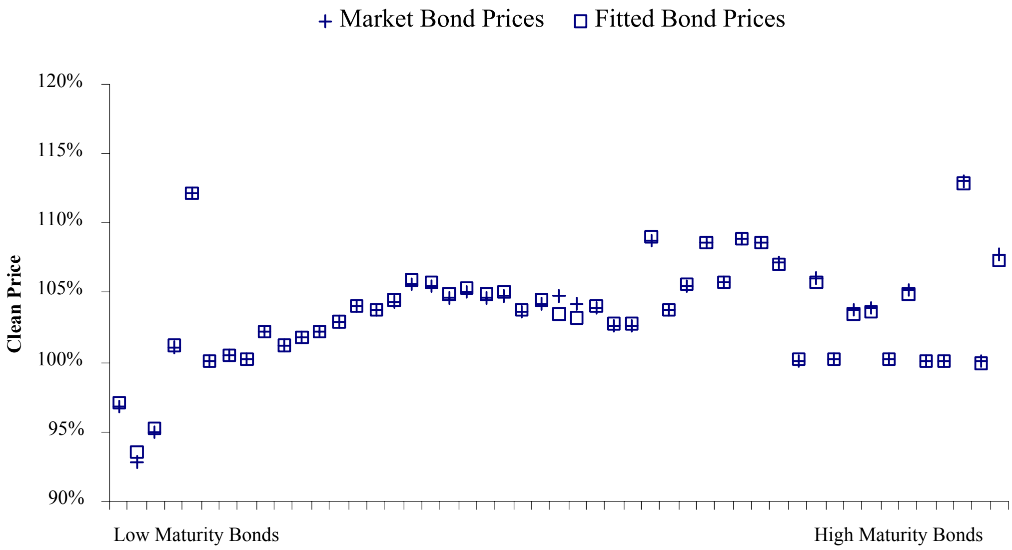

Figure 4: Estimated spot yield surface for maturities between one and 15 years

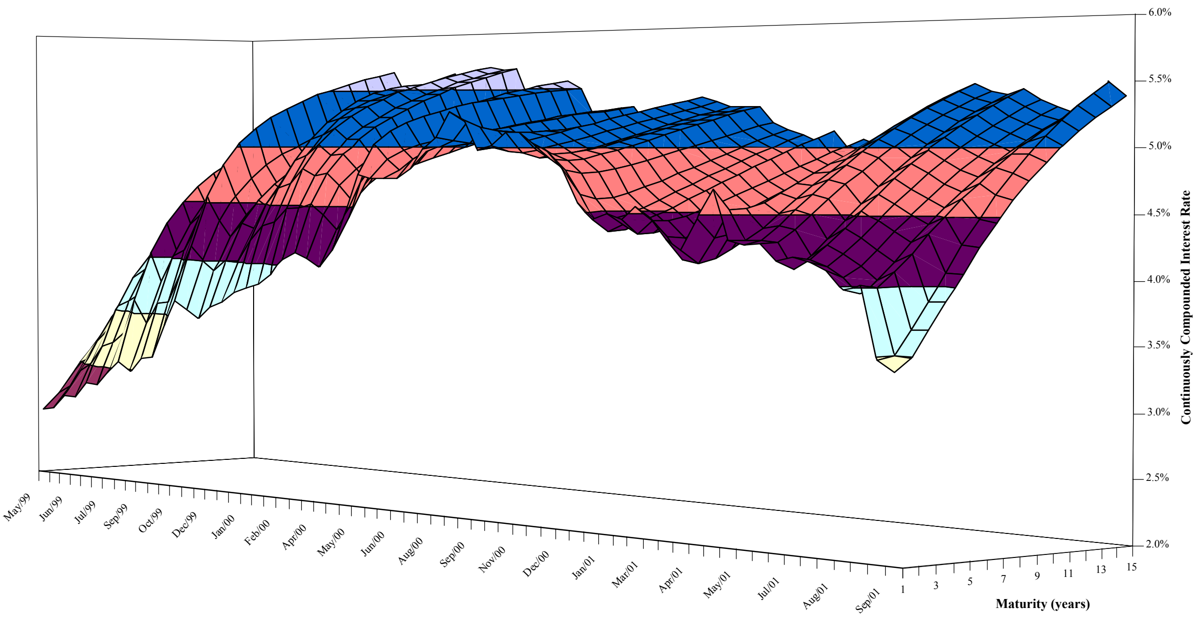

Figure 5: In-sample mean absolute percentage errors (MAPE) for the EUREX Treasury bond futures market (zero volume contracts excluded) - 4/May/99 to 28/Sep/01

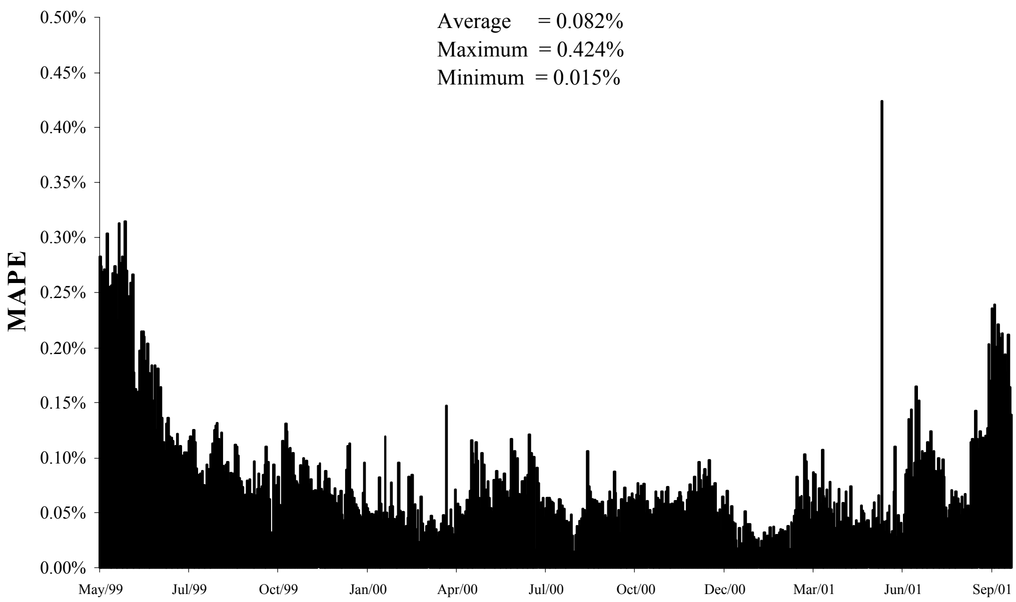

Figure 6: Time-decay pattern of the open interest in EUREX' Treasury bond futures contracts

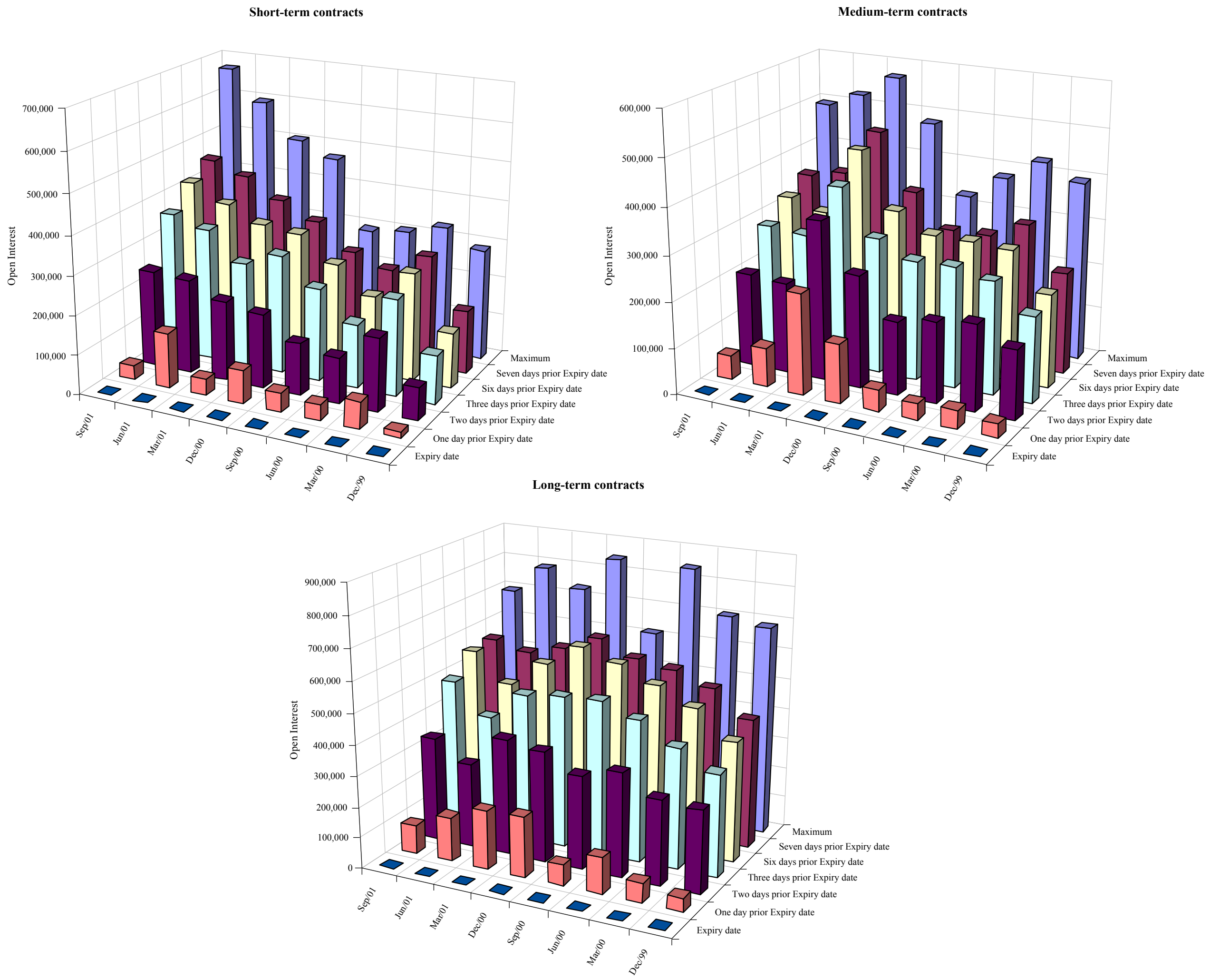

Figure 7: Average embedded quality option value by time to expiration (zero volume contracts excluded)